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Jehan Oh

Dissertation Director : Professor Sun-Sig Byun

Department of Mathematical Sciences

Seoul National University

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Abstract

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Jehan Oh

Department of Mathematical Sciences
The Graduate School
Seoul National University

In this thesis, we study non-uniformly elliptic and parabolic problems. We first consider asymptotically regular problems and establish global Calderón-Zygmund theory for the following six equations: asymptotically regular elliptic equations, asymptotically regular parabolic equations, asymptotically regular elliptic equations with irregular obstacles, asymptotically regular equations with variable growth, asymptotically linear elliptic equations in nondivergence form, and asymptotically fully nonlinear elliptic equations.

We also deal with double phase problems in divergence form on bounded nonsmooth domains. The double phase problem is characterized by the fact that its ellipticity and growth change according to the modulating coefficient, which provides a model for describing a feature of strongly anisotropic materials. We obtain global gradient estimates for distributional solutions to double phase problems with polynomial growth and with logarithmic growth.

Lastly, we investigate an optimal condition on the modulating coefficient to establish low regularity results for quasi-minimizers of the generalized double phase functional.

Key words: asymptotically regular problem, Calderón-Zygmund theory, double phase problem, non-standard growth, non-uniform ellipticity, regularity
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Contents

Chapter 1

Introduction

This thesis is concerned with regularity theory for asymptotically regular problems and double phase problems. The main feature of these problems is that their operators are non-uniformly elliptic. Non-uniformly elliptic problems are characterized by the fact that ellipticity ratio between the largest and the smallest eigenvalue of the operator is unbounded.

The first purpose of this thesis is to prove global Calderón-Zygmund theory for asymptotically regular elliptic and parabolic problems. More precisely, we are interested in an optimal Calderón-Zygmund theory for a weak solution $u \in W_0^{1,p}(\Omega)$ to the nonhomogeneous asymptotically regular elliptic problem in divergence form

$$\operatorname{div} \mathbf{a}(Du, x) = \operatorname{div} (|F|^{p-2}F) \quad \text{in} \quad \Omega, \tag{1.1}$$

and for a weak solution $u \in C^0([0,T]; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))$ to the nonhomogeneous asymptotically regular parabolic problem in divergence form

$$u_t - \operatorname{div} \mathbf{a}(Du, x, t) = \operatorname{div} (|F|^{p-2}F) \quad \text{in} \quad \Omega_T := \Omega \times (0,T), \tag{1.2}$$

with a discontinuous nonlinearity $\mathbf{a}$ and in a bounded domain Ω whose boundary $\partial\Omega$ is nonsmooth. We wish to find an affirmative answer as to what are both the weakest regularity requirement on $\mathbf{a}$ and the lowest level of geometric assumption on $\partial\Omega$ under which the relation

$$|F|^p \in L^q(\Omega) \implies |Du|^p \in L^q(\Omega), \tag{1.3}$$

for the elliptic problem, and

$$|F|^p \in L^q(\Omega_T) \implies |Du|^p \in L^q(\Omega_T), \tag{1.4}$$

for the parabolic problem, hold true for every $q \in [1, \infty)$.

Specially, if $\mathbf{a}(\xi, x)$ and $\mathbf{a}(\xi, x, t)$ are regular, see Definition 3.1.1 and 3.1.7, then the relation (1.3) and (1.4) hold true respectively under a suitable geometric assumption on $\partial\Omega$. In fact, there have been a lot of research activities about the Calderón-Zygmund theory for regular elliptic and parabolic problems, see [25, 27, 28, 29, 30, 31, 95, 93] and references therein. To extend these results, we only assume that $\mathbf{a}$ is asymptotically regular, that is, it has a more general kind of elliptic behavior near infinity, see Definition 3.1.2 and 3.1.8.

The main point of these problems is that $\mathbf{a}$ is asymptotically regular. The notion of asymptotically regular problems was first introduced in the elliptic framework by Chipot and Evans [37], and asymptotically regular problems with p-growth was considered in [107]. A local Calderón-Zygmund theory and partial Lipschitz regularity for asymptotically regular elliptic systems and minimizers have been developed by Scheven and Schmidt, see [112, 113]. Furthermore, global continuity and Lipschitz regularity results were obtained by Foss, see [62], and global Morrey regularity results were established by Byun and Oh, see [18]. As for the parabolic case, the study of asymptotically regular problems has been started, see [82]. Here we want to establish the global Calderón-Zygmund theory by proving that the gradient of a weak solution is as integrable as the nonhomogeneous term under a BMO smallness of the nonlinearity, a sufficient asymptoticity of regularity, and a sufficient flatness of the boundary in the Reifenberg sense.

Since the Calderón-Zygmund theory was first proved for the Poisson equation in [35], its nonlinear versions have been widely developed, see [31, 97, 98, 104], and up to the nonsmooth boundaries, see [25, 30, 49, 102, 103] and references therein. There are many well-known results about Calderón-Zygmund theory for regular problems, see, for instance, [23, 26, 31]. But its asymptotically regular versions are still less known. Here we want to find an asymptotically regular version for the global Calderón-Zygmund theory for the problems (1.1) and (1.2).

We would like to emphasize that the domain under consideration is the so-called Reifenberg flat domain. The boundary of the Reifenberg flat domain is so rough that even the unit normal vector can not be well defined there.

But its boundary is well trapped by two hyperplanes at both sides, inside and outside of the domain, for every point and every scale chosen. We refer to [28, 70, 108, 118] and references therein, about a concept of Reifenberg domains and their applications in analysis and geometry.

We also concerned with a global Calderón-Zygmund estimate for an asymptotically regular problem with an irregular obstacle in a bounded nonsmooth domain (see [15] for more details). The obstacle problem deals with partial differential equations in given constrained situations, the so-called obstacles. Areas of its various applications include computer science, physics, economics, biology, engineering and so on, see [11, 63, 74, 100, 109] and references therein.

In addition, we consider asymptotically regular problem with nonstandard growth. Partial differential equations with nonstandard growth have been an active and vibrant field of research in recent years. At the same time, there have been extensive research activities on the corresponding function spaces with variable exponents. These variable exponent Sobolev spaces come up when considering a number of models from mathematical physics such as electro-rheological fluids, see [105, 110, 111], fluids with temperature-dependent viscosity, see [128], porous medium, see [6, 87], homogenization of strongly anisotropic material, see [125, 130], and image restoration, see [36]. Motivated from the recent studies [1, 2, 3, 22] in non-Newtonian fluid mechanics and elliptic problems with discontinuous nonlinearities in nonsmooth domains, we are concerned with a global Calderón-Zygmund type estimate in the setting of variable exponent Lebesgue and Sobolev spaces. More precisely, we consider the following nonhomogenous asymptotically regular problem of $p(x)$-Laplacian type

$$\begin{cases} \operatorname{div} \mathbf{a}(Du, x) &= \operatorname{div} \left(|F|^{p(x)-2}F\right) \quad \text{in} \quad \Omega, \\ \qquad\quad u &= 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

with a discontinuous nonlinearity $\mathbf{a}$ and a given F. Here, Ω is a nonsmooth bounded domain in $\mathbb{R}^n$ with $n \geq 2$, and $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ is a continuous function satisfying

$$1 < \gamma_1 \leq p(x) \leq \gamma_2 < \infty$$

for some constants γ_1 and γ_2. Our goal is to establish a global Calderón-Zygmund estimate in the setting of variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ under possibly optimal conditions on the nonlinearity $\mathbf{a}$ and the boundary of

Ω by essentially proving that

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \implies |Du|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \tag{1.6}$$

holds true for $q(\cdot) : \Omega \to (1, \infty)$ satisfying

$$1 < \gamma_3 \leq q(x) \leq \gamma_4 < \infty$$

for some constants γ_3 and γ_4.

If the exponents $p(x)$ and $q(x)$ are constant functions, say $p(x) \equiv p \in (1, \infty)$ and $q(x) \equiv q \in (1, \infty)$, then there have been a lot of research activities about the Calderón-Zygmund theory for regular problems, see [26, 27, 29, 30, 76, 93, 94, 97, 98, 102] and references therein. Moreover, when the nonlinearity $\mathbf{a}$ is asymptotically regular, that is, it has a more general kind of elliptic behavior near the infinity with respect to the gradient variable, a global gradient estimate like (1.6) is still available for the case $p(x) \equiv p \in (1, \infty)$ and $q(x) \equiv q \in (1, \infty)$, see [20]. In addition, a number of researchers have considered the regularity theory for asymptotically regular problems with p-growth, see for instance [13, 37, 62, 65, 82, 107, 112, 113].

We have two main points of this problem. One is that both the exponents $p(x)$ and $q(x)$ are not constant functions. The other is that the nonlinearity $\mathbf{a}(\xi, x)$ is asymptotically regular. If $\mathbf{a}(\xi, x)$ is regular with $p(x)$-growth, see Definition 3.3.1, then the relation (1.6) holds true under a small BMO assumption in the variable x on $\mathbf{a}(\xi, x)$ and a suitable geometric assumption on $\partial\Omega$, see [22]. However, there has not been any result for asymptotically regular problems of $p(x)$-Laplacian type, as far as we are concerned in the literature. Here we employ a method of transformation introduced in [20], and partially modify this approach in the setting of variable exponent Lebesgue and Sobolev spaces. In the process, we find a suitable regular problem with $p(x)$-growth, in order to utilize an existing theory from [22], to be finally able to obtain the required estimate for the asymptotically regular problem of $p(x)$-Laplacian type, which was announced in [19].

We also study the global $W^{2,p}$ estimate for viscosity solutions to fully nonlinear, asymptotically elliptic equations of the form

$$F(D^2u, x) = f \quad \text{in } \Omega, \tag{1.7}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. It is well known that if the fully nonlinear operator $F(M, x)$ is uniformly elliptic, see Definition 3.4.1, then interior $W^{2,p}$ estimates are obtained for $n < p < \infty$ under a small oscillation of $F(M,x)$ in the variable x and $C^{1,1}$ estimates for the homogeneous equations with constant coefficients $F(D^2w, x_0) = 0$, as follows from [32, Theorem 1]. When the boundary of the domain is additionally smooth enough, namely $C^{1,1}$, the global $W^{2,p}$ estimate is obtained for $n < p < \infty$, see [122]. Those estimates in [32, 122] were extended in [52] to the range of exponents $n - \varepsilon_0 < p < \infty$ with $\varepsilon_0 > 0$ depending only on the dimension n and the ellipticity constants.

As far as we are concerned in the literature, little is known on $W^{2,p}$ regularity for asymptotically elliptic problems. For the fully nonlinear elliptic equations, the convexity of the operator with respect to the second order derivatives is crucial for the existence of solutions and the related regularity issues. Since asymptotically elliptic operators generally do not have the convexity with respect to the second order derivatives, we can not generally ensure the existence of solutions, needless to say, the uniqueness of solutions are little known. Nevertheless, we present the global $W^{2,p}$ estimates for solutions to both asymptotically linear equations and asymptotically fully nonlinear elliptic equations. These results were announced in [21].

We would like to point out that asymptotically elliptic operators are generally neither uniformly elliptic nor convex which causes a main difficulty in proving the global $W^{2,p}$ estimate. Here we transform a given asymptotically elliptic equation to a suitable uniformly elliptic equation, see Subsection 3.4.2, and then apply the existing theory for uniformly elliptic equations, see Subsection 3.4.4. In the case of an asymptotically linear operator, we first show the interior and boundary $C^{1,1}$ estimates for the limiting problem, see Lemma 3.4.10, and then establish the global $W^{2,p}$ estimate for a viscosity solution to an asymptotically linear equation in a bounded domain.

Our second purpose is to obtain global gradient estimates for double phase problems. We first consider the following equation:

$$\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F\right), \quad (1.8)$$

where $F : \Omega \to \mathbb{R}^n$ is a given vector field and $\Omega \subset \mathbb{R}^n$ is a bounded open domain with $n \geq 2$. We shall assume that the numbers p, q and the coefficient function $a : \Omega \to \mathbb{R}$ satisfy

$$1 < p < q, \quad 0 \leq a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0,1]. \quad (1.9)$$

In particular, it is of our interest to investigate sharp conditions on p, q, α and a minimal geometric assumption on $\partial\Omega$ under which the natural Calderón-Zygmund type relation

$$|F|^p + a(x)|F|^q \in L^\gamma(\Omega) \implies |Du|^p + a(x)|Du|^q \in L^\gamma(\Omega). \tag{1.10}$$

holds true for every $\gamma \in [1, \infty)$.

The equation (1.8) arises from the Euler-Lagrange equation of the functional

$$v \mapsto \mathcal{H}_{p,q}(v, \Omega) - \int_\Omega \langle |F|^{p-2}F + a(x)|F|^{q-2}F, Dv\rangle \, dx, \tag{1.11}$$

where the energy functional $\mathcal{H}_{p,q}$ is given by

$$\mathcal{H}_{p,q}(v, \Omega) := \int_\Omega \left(\frac{|Dv|^p}{p} + a(x)\frac{|Dv|^q}{q} \right) dx. \tag{1.12}$$

Recently there has been an increasing interest in the functional $\mathcal{H}_{p,q}$. This functional was first introduced by Zhikov [125, 130] for the purpose of describing a feature of strongly anisotropic materials with hardening exponents p and q. In the functional $\mathcal{H}_{p,q}$, the modulating coefficient $a(\cdot)$ plays a role in dictating the geometry of the mixture of two different materials. On the points x with $a(x) > 0$ the functional $\mathcal{H}_{p,q}$ has q-growth in the gradient, while on the zero set $\{a(x) = 0\}$ it has p-growth. In this respect, this functional exhibits a dramatic change of its growth around the zero set $\{a(x) = 0\}$. Indeed, when $\frac{q}{p} > 1 + \frac{\alpha}{n}$, the Lavrentiev phenomenon appears and it is also possible to construct a functional as in (1.12) which admits a minimizer such that its singular set has a Hausdorff dimension arbitrarily close to $n - p$, as we have seen from the earlier works [55, 61]. To exclude this irregularity, we consider a distributional solutions to the problem (1.8) under the main assumption

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{1.13}$$

It is worth pointing out that the equation which we are considering is non-uniformly elliptic. This means that if we let

$$A(x, \xi) := |\xi|^{p-2}\xi + a(x)|\xi|^{q-2}\xi, \tag{1.14}$$

if we denote $D_\xi A(x,\xi)$ to mean the Jacobian matrix of $A(x,\xi)$ with respect to the gradient variable ξ, and if $a(x_0)=0$ for some $x_0 \in B_R(y) \subset \Omega$, then

$$\frac{\text{the highest eigenvalue of } D_\xi A(x,\xi)}{\text{the lowest eigenvalue of } D_\xi A(x,\xi)},$$

which can be controlled by $1+R^\alpha|\xi|^{q-p}$, and so the above ratio is unbounded with respect to the gradient variable ξ. In past years, there have been many research activities regarding non-uniformly elliptic operators, see [53, 54, 55, 84, 117, 119, 120, 125, 127, 129, 130] and references therein. An important point to be noted is that they are naturally related to non-standard growth conditions.

In the case $a(\cdot) \equiv 0$, the problem (1.8) becomes a classical p-Laplacian type one which is uniformly elliptic and has standard growth conditions. In this case, there have been a lot of regularity results of the Calderón-Zygmund theory, see [31, 35, 43, 71, 76] for the interior estimates, and [29, 30, 73, 95, 97, 98, 102] for the estimates up to boundary, respectively. For the general case, there have been also a number of intensive studies during the last years, see [7, 8, 38, 39, 55, 61]. Especially, we would like to mention the pioneering work of Colombo and Mingione [40], who established the local Calderón-Zygmund type estimates for a class of non-uniformly elliptic problems. Here we provide a Calderón-Zygmund type estimate up to the boundary, which was announced in [17].

We next consider the borderline case of double phase problems in divergence form. The problem under consideration is given by

$$\begin{aligned} &\operatorname{div}\left(\beta(x)\left[|Du|^{p-2}Du + a(x)|Du|^{p-2}\log(e+|Du|)Du\right]\right) \\ &\qquad = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{p-2}\log(e+|F|)F\right) \quad \text{in } \Omega, \end{aligned} \tag{1.15}$$

where $1<p<\infty$ is a fixed number, $\Omega \subset \mathbb{R}^n$ is a bounded open domain with $n \geq 2$, and $F : \Omega \to \mathbb{R}^n$ is a given vector field. Here the coefficient function $\beta : \mathbb{R}^n \to \mathbb{R}$ satisfies $\nu \leq \beta(\cdot) \leq L$ for some fixed constants $0 < \nu \leq L < \infty$, and the function $a : \mathbb{R}^n \to \mathbb{R}$ is always assumed to be non-negative and bounded. We investigate optimal conditions on the coefficient functions $\beta(\cdot), a(\cdot)$ and a minimal geometric assumption on $\partial\Omega$ under which the natural Calderón-Zygmund type relation

$$\begin{aligned} &|F|^p + a(x)|F|^p\log(e+|F|) \in L^\gamma(\Omega) \\ &\qquad \Longrightarrow |Du|^p + a(x)|Du|^p\log(e+|Du|) \in L^\gamma(\Omega) \end{aligned} \tag{1.16}$$

hold true for every $\gamma \in [1,\infty)$.

We remark that the equation (1.15) is closely related to the functional

$$v \in W^{1,1}(\Omega) \ \mapsto \ \mathcal{H}(v,\Omega) - \int_\Omega \langle |F|^{p-2}F + a(x)|F|^{p-2}\log(e+|F|)F, Dv\rangle\,dx, \tag{1.17}$$

where the energy functional $\mathcal{H}$ is given by

$$\mathcal{H}(v,\Omega) := \int_\Omega \beta(x)\left[|Dv|^p + a(x)|Dv|^p\log(e+|Dv|)\right]dx. \tag{1.18}$$

It is worth pointing out that this energy functional $\mathcal{H}$ can be regarded as a borderline case of (p,q)-energy functionals

$$\mathcal{H}_{p,q}(v,\Omega) = \int_\Omega \beta(x)\left[|Dv|^p + a(x)|Dv|^q\right]dx, \quad 1<p<q. \tag{1.19}$$

As before, we observe that the regularity of the modulating coefficient $a(\cdot)$ is closely related to how to control the size of the phase transition. In the borderline case, $a(\cdot)$ is not necessary to be Hölder continuous, but it seems that the correct condition is to be log-Hölder continuous. The point is that the log-Hölder continuity of $a(\cdot)$ is exactly dual to the size of the phase transition in $\mathcal{H}$.

In particular, we consider discontinuous coefficients of bounded mean oscillations (BMO) and nonsmooth domains which are not necessarily Lipschitz continuous. To deal with logarithmic terms in the functional (1.17) and conduct a careful analysis near the nonsmooth boundary, we utilize the framework of Orlicz spaces and Musielak-Orlicz spaces. In addition, we adopt the so-called maximal function free technique which was first introduced in [4] and later utilized in [11, 24, 26, 51, 123, 124]. The main results for this problem were announced in [16].

Lastly, we consider the double phase functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{F}(v,\Omega) := \int_\Omega \left[G(|Dv|) + a(x)H(|Dv|)\right]dx, \tag{1.20}$$

where $G, H : [0,\infty) \to [0,\infty)$ are Young functions. The main feature of the functional (1.20) is that the energy density changes its growth and ellipticity properties according to the modulating coefficient $a(\cdot)$. The double phase functional (1.20) is a natural generalization of the one with (p, q)-type

$$v \in W^{1,1}(\Omega) \mapsto \int_{\Omega} [|Dv|^p + a(x)|Dv|^q] \, dx, \quad q > p > 1, \tag{1.21}$$

and the one in a borderline case

$$v \in W^{1,1}(\Omega) \mapsto \int_{\Omega} [|Dv|^p + a(x)|Dv|^p \log(1 + |Dv|)] \, dx, \quad p > 1. \tag{1.22}$$

Zhikov [125, 130] first introduced a family of functionals including (1.21) for the purpose of describing a feature of strongly anisotropic materials: the modulating coefficient $a(\cdot)$ presents the geometry of the mixture of two different materials. As shown in [55, 61, 127, 129], such functionals exhibit Lavrentiev phenomenon whereby minimizers are irregular and even discontinuous. On the other hand, the functionals (1.21) and (1.22) belong to the class of functionals having (p, q)-growth condition. These are functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \int_{\Omega} F(x, Dv) \, dx,$$

where the energy density $F(x, \xi)$ satisfies

$$|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q + 1, \quad q > p > 1.$$

This (p, q)-growth condition has been first treated by Marcellini [89, 90, 91] and extensively studied in recent years, see [12, 53, 54, 55, 61, 64, 114, 115] and references therein.

In the case $p > n$, it is clear from the Sobolev embedding theorem that quasi-minimizers of the functionals (1.21) and (1.22) are locally bounded and Hölder continuous. Very recently, Baroni, Colombo and Mingione [7, 38, 39] found that when $p \leq n$, the optimal condition for Hölder continuity of quasi-minimizers of the functional (1.21) is $a(\cdot) \in C^{0,\alpha}(\Omega)$ with $\alpha \in (0, 1]$ and $q \leq p + \alpha$. Furthermore, for the functional (1.22), the log-Hölder continuity of $a(\cdot)$ is sufficient in order to obtain the Hölder continuity of quasi-minimizers, see [7, 8]. These results show that the regularity of the modulating coefficient $a(\cdot)$ is closely related to how to control the size of the associated phase transition.

We investigate an optimal condition on the modulating coefficient $a(\cdot)$ in the functional (1.20) under which the Hölder regularity result holds for local quasi-minimizers. We provide a reasonable condition on the modulus of continuity of $a(\cdot)$, see (5.44), and prove local boundedness, Hölder continuity via De Giorgi's method and the Harnack inequality under this condition. We remark that our condition agrees with the known one in the classical case, see Remark 5.2.2, and serves the natural assumption for the modulating coefficient in a wide variety of double phase functionals such as

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega \left[|Dv|^p + a(x)|Dv|^p \left[\log(1+|Dv|)\right]^\gamma\right] dx, \quad p > 1,\ \gamma > 0,$$

and

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega \left[|Dv|^p + a(x)|Dv|^p \log\log(e+|Dv|)\right] dx, \quad p > 1,$$

see Remark 5.3.12. In addition, we present a sharp condition for the absence of the Lavrentiev phenomenon, see Theorem 5.2.1.

The rest of the thesis is organized as follows. In the next chapter, we introduce some notations, function spaces, and analytic tools. In Chapter 3, we prove global Calderón-Zygmund theory for asymptotically regular problems. Chapter 4 is devoted to the gradient estimates for double phase problems. In Chapter 5, we investigate an optimal condition on the modulating coefficient to establish low regularity results for the generalized double phase functional.

Chapter 2

Preliminaries

2.1 Notations

Throughout the thesis, we will use standard notations and will assume that the functions and sets considered are measurable.

1. For a point $y \in \mathbb{R}^n$ and for $r > 0$, $B_r(y) := \{x \in \mathbb{R}^n : |x - y| < r\}$, $\Omega_r(y) := B_r(y) \cap \Omega$, $\partial_w \Omega_r(y) := B_r(y) \cap \partial\Omega$. If the center is clear in the context, we shall omit denoting it as follows : $B_r \equiv B_r(y)$, $\Omega_r \equiv \Omega_r(y)$.

2. $B_r^+ = B_r(0) \cap \{x_n > 0\}$, $T_r = B_r(0) \cap \{x_n = 0\}$.

3. For a point $(y, s) \in \mathbb{R}^n \times \mathbb{R}$ and positive numbers ρ, θ, the parabolic cylinder under consideration is
$$Q_{(\rho,\theta)}(y, s) = B_\rho(y) \times (s - \theta, s + \theta).$$

4. For a set $\mathcal{S} \subset \mathbb{R}^n$, with $a(\cdot)$ being the coefficient which we consider, we shall denote
$$[a]_{C^{0,\alpha}(\mathcal{S})} := \sup_{x,y\in\mathcal{S}, x\neq y} \frac{|a(x) - a(y)|}{|x - y|^\alpha}.$$
In the case $\mathcal{S} = \Omega$, we shall use the abbreviated notation $[a]_{C^{0,\alpha}} \equiv [a]_{C^{0,\alpha}(\Omega)}$ and $\|a\|_{L^\infty} \equiv \|a\|_{L^\infty(\Omega)}$.

5. For a vector valued function $X : \mathbb{R}^n \to \mathbb{R}^m$, a vector $e \in B_1 \subset \mathbb{R}^n$ and

a real number $h \in \mathbb{R}$, we define the finite difference operator

$$\tau_{e,h}X(x) \equiv (\tau_{e,h}X)(x) := X(x+he) - X(x).$$

If $e = e_s$ with $s \in \{1, \cdots, n\}$, then we write $\tau_{s,h}$ instead of $\tau_{e,h}$.

6. The number e is the Euler's number and $\log t$ is the natural logarithm of $t > 0$.

7. For $\alpha > 0$, $\log^{\alpha}(e+t)$ denotes the quantity $[\log(e+t)]^{\alpha}$.

8. Let us denote by $(f)_U$ to mean the integral average of a locally integrable function $f = f(x)$ over a bounded domain U in $\mathbb{R}^n$, that is,

$$(f)_U = \fint_U f(x)\,dx = \frac{1}{|U|}\int_U f(x)\,dx.$$

9. For a function v, we write $v_{\pm} := \max\{\pm v, 0\}$.

2.2 Function spaces

2.2.1 Variable exponent Lebesgue and Sobolev spaces

In this subsection, we present the generalized Lebesgue-Sobolev space involving variable exponent $p(\cdot)$. Given a bounded domain $U \subset \mathbb{R}^n$ and a bounded measurable function $p = p(\cdot) : U \subset \mathbb{R}^n \to (1, \infty)$, the variable exponent Lebesgue space $L^{p(\cdot)}(U;\mathbb{R}^N)$, $N \geq 1$, consists of all measurable functions $f : U \to \mathbb{R}^N$ such that

$$\int_U |f(x)|^{p(x)} dx < +\infty,$$

with the following Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(U;\mathbb{R}^N)} := \inf\left\{ t > 0 : \int_U \left|\frac{f(x)}{t}\right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(U;\mathbb{R}^N)$ is a collection of all measurable functions $f : U \to \mathbb{R}^N$ such that f is weakly differentiable and its gradient Df belongs to $L^{p(\cdot)}(U;\mathbb{R}^{Nn})$, that is

$$W^{1,p(\cdot)}(U;\mathbb{R}^N) := \left\{ f \in L^{p(\cdot)}(U;\mathbb{R}^N) : Df \in L^{p(\cdot)}(U;\mathbb{R}^{Nn}) \right\}.$$

Furthermore, the $W^{1,p(\cdot)}$-norm of f is defined by

$$\|f\|_{W^{1,p(\cdot)}(U;\mathbb{R}^N)} := \|f\|_{L^{p(\cdot)}(U;\mathbb{R}^N)} + \|Df\|_{L^{p(\cdot)}(U;\mathbb{R}^{Nn})}.$$

We denote $W_0^{1,p(\cdot)}(U;\mathbb{R}^N)$ by the closure of $C_0^\infty(U;\mathbb{R}^N)$ in $W^{1,p(\cdot)}(U;\mathbb{R}^N)$. Observe that $W^{1,p(\cdot)}(U;\mathbb{R}^N)$ and $W_0^{1,p(\cdot)}(U;\mathbb{R}^N)$ are separable reflexive Banach spaces. For $N=1$, we simply write $L^{p(\cdot)}(U)$, $W^{1,p(\cdot)}(U)$ and $W_0^{1,p(\cdot)}(U)$.

For the study of partial differential equations on $L^{p(\cdot)}$ spaces, we usually impose a certain regularity condition on the variable exponent $p(\cdot)$. This basic regularity condition is the so-called log-Hölder continuity. We say that $p(\cdot)$ is log-Hölder continuous in U if there exists a constant $L>0$ such that

$$|p(x)-p(y)| \leq \frac{L}{\log\left(e + \frac{1}{|x-y|}\right)},$$

for all $x,y \in U$. We remark that, $p(\cdot)$ is log-Hölder continuous if and only if $p(\cdot)$ admits a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ satisfying

$$\limsup_{r\to 0} \omega(r) \log\left(\frac{1}{r}\right) < +\infty.$$

The log-Hölder continuity condition enables us to employ many crucial tools in classical analysis. For example, with this condition, the Hardy-Littlewood maximal operator is bounded within the framework of variable exponent Lebesgue spaces. Also the Sobolev embedding and the Poincaré inequality hold in variable exponent Sobolev spaces. For further properties about variable exponent spaces with log-Hölder continuous exponents, we refer to [45, 46, 47, 48, 68, 80] and references therein.

2.2.2 Orlicz spaces and Musielak-Orlicz spaces

We first recall some definitions and basic properties on the Orlicz spaces. A Young function $g : [0,\infty) \to [0,\infty)$ is a nondecreasing convex function such that

$$g(0)=0,\ \lim_{t\to\infty} g(t)=\infty,\ \lim_{t\to 0+}\frac{g(t)}{t}=0,\ \lim_{t\to\infty}\frac{g(t)}{t}=\infty.$$

For a given Young function g, the complementary Young function g^* to g is given by

$$g^*(t)=\sup\{ts-g(s): s\geq 0\},\ t\geq 0.$$

We remark that this g^* satisfies all the conditions to be a Young function and that $(g^*)^*=g$.

We say that g satisfies the Δ_2-condition, denoted by $g\in\Delta_2$, if there exists $\mu_1>1$ such that $g(2t)\leq \mu_1 g(t)$ for all $t\geq 0$. We denote by $\Delta_2(g)$ the smallest constant μ_1. Also we say that g satisfies the ∇_2-condition, denoted by $g\in\nabla_2$, if there exists $\mu_2>1$ such that $g(t)\leq \frac{1}{2\mu_2}g(\mu_2 t)$ for all $t\geq 0$. We note that $g\in\nabla_2$ if and only if $g^*\in\Delta_2$.

If $g\in\Delta_2$, then there exist two constants $\kappa_1=\kappa_1(\Delta_2(g))$ and $\kappa_2=\kappa_2(\Delta_2(g))$ with $1<\kappa_1\leq\kappa_2<\infty$ such that

$$c^{-1}\min\{\theta^{\kappa_1},\theta^{\kappa_2}\}g(t)\leq g(\theta t)\leq c\max\{\theta^{\kappa_1},\theta^{\kappa_2}\}g(t)\quad \text{for all } t,\theta\geq 0, \tag{2.1}$$

where the constant $c\geq 1$ is independent of θ and t (see [79]).

If $g\in\Delta_2\cap\nabla_2$, then for any $\varepsilon\in(0,1]$, there exists a positive constant c depending on ε, $\Delta_2(g)$ and $\Delta_2(g^*)$ such that

$$st\leq \varepsilon g(s)+cg^*(t)\quad \text{for all } s,t\geq 0. \tag{2.2}$$

This inequality is called Young's inequality. Furthermore, from the following property of the complementary Young function (see for instance [5])

$$g^*\left(\frac{g(t)}{t}\right)\leq g(t), \tag{2.3}$$

we obtain a modified form of Young's inequality:

$$s\frac{g(t)}{t}\leq \varepsilon g(s)+cg(t)\quad \text{for all } s\geq 0 \text{ and } t>0. \tag{2.4}$$

In particular, when $g(t)=t^p\log(e+t)$ with $p>1$, we have

$$st^{p-1}\log(e+t)\leq \varepsilon s^p\log(e+s)+ct^p\log(e+t)\quad \text{for all } s,t\geq 0. \tag{2.5}$$

For a Young function g, the Orlicz class $K^g(\Omega)$ consists of all measurable functions $v : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega g(|v(x)|)\, dx < +\infty.$$

The Orlicz space $L^g(\Omega)$ is the vector space generated by $K^g(\Omega)$. If $g \in \Delta_2$, then $K^g(\Omega) = L^g(\Omega)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^g(\Omega)} = \inf\left\{\sigma > 0 : \fint_\Omega g\left(\frac{|v(x)|}{\sigma}\right) dx \leq 1\right\}.$$

If $g \in \Delta_2 \cap \nabla_2$, then for any $v \in L^g(\Omega)$ and $w \in L^{g^*}(\Omega)$,

$$\fint_\Omega |vw|\, dx \leq 2\, \|v\|_{L^g(\Omega)}\, \|w\|_{L^{g^*}(\Omega)}\,. \tag{2.6}$$

The followings are some examples of Young functions and the corresponding Orlicz spaces:

$$g_1(t) = t^p,\ \ p \geq 1,$$

which generates the classical Lebesgue space $L^p(\Omega)$, and

$$g_2(t) = t^p \log^\alpha(e+t),\ \ p \geq 1,\ \ \alpha > 0,$$

generates the Orlicz-Zygmund space $L^p \log^\alpha L(\Omega)$. Above all, $L\log^\alpha L$ space includes any L^p spaces for $p > 1$ with the following estimate:

$$\|v\|_{L\log^\alpha L(\Omega)} \leq c(p)\left(\fint_\Omega |v|^p\, dx\right)^{\frac{1}{p}}.$$

Moreover, for every $p > 1$ and $v \in L^p(\Omega)$, we have that (see [3, 72] for more details)

$$\fint_\Omega |v| \log^\alpha\left(e + \frac{|v|}{(|v|)_\Omega}\right) dx \leq c(p,\alpha)\left(\fint_\Omega |v|^p\, dx\right)^{\frac{1}{p}}. \tag{2.7}$$

We now introduce the Musielak-Orlicz spaces which generalize the Orlicz spaces. Let $g : \Omega \times [0,\infty) \to [0,\infty)$ be a function satisfying the following conditions:

(a) $g(x,\cdot)$ is a Young function for every $x \in \Omega$,

(b) $g(\cdot,t)$ is a measurable function for every $t \geq 0$.

Such a function $g(x,t)$ is called a Musielak-Orlicz function. We will denote by $g(\cdot)$ a Musielak-Orlicz function to emphasize the dependence on x. As before, we say that $g(\cdot)$ satisfies the Δ_2-condition, denoted by $g(\cdot) \in \Delta_2$, if there exists $\mu > 1$ such that $g(x,2t) \leq \mu g(x,t)$ for all $x \in \Omega,\ t \geq 0$.

The Musielak-Orlicz class $K^{g(\cdot)}(\Omega)$ consists of all measurable functions $v : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega g(x,|v(x)|)\,dx < +\infty,$$

and the Musielak-Orlicz space $L^{g(\cdot)}(\Omega)$ is the vector space generated by $K^{g(\cdot)}(\Omega)$. If $g(\cdot) \in \Delta_2$, then $K^{g(\cdot)}(\Omega) = L^{g(\cdot)}(\Omega)$ and this space is a Banach space under the Luxemburg type norm

$$\|v\|_{L^{g(\cdot)}(\Omega)} = \inf\left\{\sigma > 0 : \fint_\Omega g\left(x, \frac{|v(x)|}{\sigma}\right)\,dx \leq 1\right\}.$$

The Musielak-Orlicz-Sobolev space $W^{1,g(\cdot)}(\Omega)$ is the function space of all measurable functions $v \in L^{g(\cdot)}(\Omega)$ such that its distributional gradient vector Dv belongs to $L^{g(\cdot)}(\Omega;\mathbb{R}^n)$. If $v \in W^{1,g(\cdot)}(\Omega)$, we define its norm to be

$$\|v\|_{W^{1,g(\cdot)}(\Omega)} = \|v\|_{L^{g(\cdot)}(\Omega)} + \|Dv\|_{L^{g(\cdot)}(\Omega;\mathbb{R}^n)}.$$

The space $W_0^{1,g(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,g(\cdot)}(\Omega)$. For a deeper discussion of the Musielak-Orlicz space and the associated Sobolev space, we refer the reader to [9, 44, 58, 59, 99, 116] and references therein.

2.3 Auxiliary lemmas

We first present some technical lemmas.

Lemma 2.3.1. *[66, 67] Let $\phi : [R_1, R_2] \to [0,\infty)$ be a bounded function. Suppose that for any s_1 and s_2 with $0 < R_1 \leq s_1 < s_2 \leq R_2$,*

$$\phi(s_1) \leq \vartheta\phi(s_2) + \frac{P}{(s_2 - s_1)^\kappa} + Q,$$

where $P, Q \geq 0$, $\kappa > 0$ and $\vartheta \in [0, 1)$. Then there holds

$$\phi(R_1) \leq c(\vartheta, \kappa) \left[\frac{P}{(R_2 - R_1)^\kappa} + Q \right].$$

Lemma 2.3.2. *[83] Let $\{Y_i\}_{i=0}^\infty$ be a sequence of nonnegative numbers satisfying the recursive inequalities*

$$Y_{i+1} \leq C b^i Y_i^{1+\delta}, \quad i = 0, 1, 2, \cdots, \tag{2.8}$$

where $C, b > 1$ and $\delta > 0$ are given numbers. If

$$Y_0 \leq C^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}, \tag{2.9}$$

then $Y_i \to 0$ as $i \to \infty$.

Lemma 2.3.3. *[83] Let $v \in W^{1,1}(B_\rho)$. For any $l > k$, we have*

$$(l - k)|B_\rho \cap \{v > l\}|^{1-\frac{1}{n}} \leq \frac{c|B_\rho|}{|B_\rho \setminus \{v > k\}|} \int_{B_\rho \cap \{k < v \leq l\}} |Dv| \, dx$$

for some positive constant c depending only on n.

Lemma 2.3.4. *[83] Let $v \in W^{1,1}(B_\rho)$. For any $l > k$, we have*

$$(l - k)|B_\rho \cap \{v < k\}|^{1-\frac{1}{n}} \leq \frac{c|B_\rho|}{|B_\rho \setminus \{v < l\}|} \int_{B_\rho \cap \{k < v \leq l\}} |Dv| \, dx$$

for some positive constant c depending only on n.

In Section 4.2, we will use the following logarithmic inequalities:

$$\log(e + st) \leq \log(e + s) + \log(e + t), \tag{2.10}$$

$$\log(e + t) \leq c(\alpha) \log(e + t^\alpha), \tag{2.11}$$

for all $s, t \geq 0$ and $\alpha \geq 1$.

Chapter 3

Global Calderón-Zygmund theory for asymptotically regular problems

3.1 $W^{1,q}$-estimates for asymptotically regular nonlinear elliptic and parabolic equations

In this section, we are concerned with an optimal Calderón-Zygmund theory for a weak solution $u \in W_0^{1,p}(\Omega)$ to the nonhomogeneous asymptotically regular elliptic problem in divergence form

$$\operatorname{div} \mathbf{a}(Du, x) = \operatorname{div} (|F|^{p-2}F) \quad \text{in} \quad \Omega, \tag{3.1}$$

and for a weak solution $u \in C^0([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ to the nonhomogeneous asymptotically regular parabolic problem in divergence form

$$u_t - \operatorname{div} \mathbf{a}(Du, x, t) = \operatorname{div} (|F|^{p-2}F) \quad \text{in} \quad \Omega_T := \Omega \times (0, T), \tag{3.2}$$

with a discontinuous nonlinearity $\mathbf{a}$ and in a bounded domain Ω whose boundary $\partial\Omega$ is nonsmooth. In particular, we wish to find an affirmative answer as to what are both the weakest regularity requirement on $\mathbf{a}$ and the lowest level of geometric assumption on $\partial\Omega$ under which the relation

$$|F|^p \in L^q(\Omega) \implies |Du|^p \in L^q(\Omega), \tag{3.3}$$

for the elliptic problem, and

$$|F|^p \in L^q(\Omega_T) \implies |Du|^p \in L^q(\Omega_T), \tag{3.4}$$

for the parabolic problem, hold true for every $q \in [1, \infty)$.

We will prove the main results, Theorems 3.1.5 and 3.1.10, by converting a given asymptotically regular problem to a suitable regular problem.

3.1.1 Hypotheses and main results

We first deal with the elliptic case. Let Ω be a bounded open domain in $\mathbb{R}^n$ with $n \geq 2$ and let $1 < p < \infty$ be a fixed real number. We then consider the following non-linear elliptic problem in divergence form:

$$\begin{cases} \text{div } \mathbf{a}(Du, x) = \text{div } (|F|^{p-2}F) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.5}$$

where $F = (f_1, \cdots, f_n) \in L^p(\Omega; \mathbb{R}^n)$.

Here, a vector-valued function

$$\mathbf{a}(\xi, x) = (a_1(\xi, x), \cdots, a_n(\xi, x)) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is assumed to be a Carathéodory function, namely, measurable in x and continuous in ξ. To introduce the definition that $\mathbf{a}(\xi, x)$ is asymptotically regular, we start with the definition that $\mathbf{b}(\xi, x)$ is regular.

Definition 3.1.1. $\mathbf{b}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is regular if* $\mathbf{b}(\xi, x)$ *is a* C^1 *function in* ξ *and there exist positive constants* λ *and* Λ *such that*

$$|\mathbf{b}(\xi, x)| + |\xi||D_\xi \mathbf{b}(\xi, x)| \leq \Lambda |\xi|^{p-1} \tag{3.6}$$

and

$$D_\xi \mathbf{b}(\xi, x)\eta \cdot \eta \geq \lambda |\xi|^{p-2} |\eta|^2 \tag{3.7}$$

for almost every $x \in \mathbb{R}^n$ *and all* $\xi, \eta \in \mathbb{R}^n$.

Here $D_\xi \mathbf{b}(\xi, x)$ denotes the Jacobian matrix of $\mathbf{b}(\xi, x)$ with respect to ξ. We remark that the above structural conditions (3.6)-(3.7) imply the following monotonicity conditions: for each $\xi, \eta \in \mathbb{R}^n$ and for almost every $x \in \mathbb{R}^n$,

$$(\mathbf{b}(\xi, x) - \mathbf{b}(\eta, x)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^p \quad \text{if} \ \ p \geq 2,$$

$$(\mathbf{b}(\xi,x)-\mathbf{b}(\eta,x))\cdot(\xi-\eta)\geq\gamma|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}\quad\text{ if }\ 1<p<2,$$

where γ is a positive constant depending only on n, p, and λ.

In this section, we are mainly concerned with the Calderón-Zygmund theory for asymptotically regular problems. To do this, we assume that $\mathbf{a}(\xi,x)$ is asymptotically δ-regular with δ being determined later.

Definition 3.1.2. $\mathbf{a}(\xi,x)$ *is asymptotically* δ*-regular if there exists a regular vector-valued function* $\mathbf{b}(\xi,x)$ *such that*

$$\limsup_{|\xi|\to\infty}\frac{|\mathbf{a}(\xi,x)-\mathbf{b}(\xi,x)|}{|\xi|^{p-1}}\leq\delta, \tag{3.8}$$

uniformly with respect to $x\in\mathbb{R}^n$*.*

We remark that (3.8) can be reduced to

$$|\mathbf{a}(\xi,x)-\mathbf{b}(\xi,x)|\leq\omega(|\xi|)(1+|\xi|^{p-1}) \tag{3.9}$$

for some uniformly bounded function $\omega:\mathbb{R}^+\to\mathbb{R}^+$ with $\limsup_{r\to\infty}\omega(r)\leq\delta$. Here, $\omega:\mathbb{R}^+\to\mathbb{R}^+$ can be defined as follows:

$$\omega(r):=\sup_{x\in\mathbb{R}^n,|\xi|=r}\frac{|\mathbf{a}(\xi,x)-\mathbf{b}(\xi,x)|}{1+r^{p-1}},\quad(r\geq0).$$

We point out that Definition 3.1.2 is weaker than the definition of asymptotic regularity in [112], as we do not require that $\omega(|\xi|)\to0$ as $|\xi|\to\infty$. For this reason, we include oscillating cases, for example, $\mathbf{a}(\xi,x)=\mathbf{a}(\xi)=|\xi|^{p-2}\xi+\delta\sin(|\xi|^2)|\xi|^{p-2}\xi$ and $\mathbf{a}(\xi,x)=|\xi|^{p-2}\xi+\delta\sin(|x||\xi|)|\xi|^{p-2}\xi$ for $\delta>0$.

In order to measure the oscillation of $\mathbf{b}(\xi,x)$ over a bounded set $B_r(y)$ in the variable x, we define

$$\Theta(\mathbf{b},B_r(y))(x)=\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}\frac{|\mathbf{b}(\xi,x)-\overline{\mathbf{b}}_{B_r(y)}(\xi)|}{|\xi|^{p-1}}, \tag{3.10}$$

where $\overline{\mathbf{b}}_{B_r(y)}(\xi)$ is the integral average of $\mathbf{b}(\xi,\cdot)$ in the variable x over $B_r(y)$ for each fixed $\xi\in\mathbb{R}^n$, as defined by

$$\overline{\mathbf{b}}_{B_r(y)}(\xi)=⨍_{B_r(y)}\mathbf{b}(\xi,x)dx=\frac{1}{|B_r(y)|}\int_{B_r(y)}\mathbf{b}(\xi,x)dx. \tag{3.11}$$

Definition 3.1.3. $\mathbf{b}(\xi, x)$ *is* (δ, R)*-vanishing if we have*

$$\sup_{0<r\leq R}\ \sup_{y\in\mathbb{R}^n} ⨍_{B_r(y)} |\Theta(\mathbf{b}, B_r(y))(x)|\ dx \leq \delta. \tag{3.12}$$

We remark that by the scaling, one can take $R \geq 1$ to be any number while a small positive constant $\delta > 0$ is still invariant under such a scaling.

Definition 3.1.4. Ω *is* (δ, R)*-Reifenberg flat if for every* $x \in \partial\Omega$ *and every* $r \in (0, R]$*, there exists a coordinate system* $\{y_1, \cdots, y_n\}$*, which can depend on* r *and* x *so that* $x = 0$ *in this coordinate system and that*

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}.$$

This geometric condition prescribes that under all scales the boundary can be trapped between two hyperplanes, depending on the scale chosen. The domain can go beyond Lipschitz category, not necessarily given by graphs.

We now state the main result of the elliptic case.

Theorem 3.1.5. *For any given* $q \in (1, \infty)$*, assume that* $F \in L^{pq}(\Omega; \mathbb{R}^n)$*. Then there exists a constant* $\delta = \delta(n, p, q, \lambda, \Lambda) > 0$ *such that if* $\mathbf{a}(\xi, x)$ *is asymptotically* δ*-regular with* $\mathbf{b}(\xi, x)$ *which is* (δ, R)*-vanishing, and if* Ω *is* (δ, R)*-Reifenberg flat, then a weak solution* $u \in W_0^{1,p}(\Omega)$ *to the problem (3.5) satisfies* $Du \in L^{pq}(\Omega; \mathbb{R}^n)$ *with the estimate*

$$\|Du\|_{L^{pq}(\Omega)} \leq c\left(\|F\|_{L^{pq}(\Omega)} + 1\right), \tag{3.13}$$

where c *is a positive constant depending on* n*,* p*,* q*,* λ*,* Λ*,* ω*, and* $|\Omega|$*.*

Remark 3.1.6. *As a consequence of Theorem 3.1.5, we have Hölder regularity. More precisely, if we take* q *with* $pq > n$*, then it follows directly that* $u \in C^{0,1-\frac{n}{pq}}(\overline{\Omega})$ *by the Sobolev inequality.*

Next, we deal with the parabolic case. For a fixed $T > 0$, write a parabolic cylinder

$$\Omega_T = \Omega \times (0, T),$$

and its parabolic boundary

$$\partial_p\Omega_T = \partial\Omega \times [0, T] \cup \Omega \times \{t = 0\}.$$

Let $\frac{2n}{n+2} < p < \infty$ be a fixed real number. We then consider the following non-linear parabolic problem in divergence form:

$$\begin{cases} u_t - \text{div } \mathbf{a}(Du, x, t) = \text{div } (|F|^{p-2}F) & \text{in} \quad \Omega_T, \\ u = 0 & \text{on} \quad \partial_p \Omega_T, \end{cases} \tag{3.14}$$

where $u = u(x, t)$ is an unknown, $u_t = \partial_t u = \partial u/\partial t$ is the time derivative of u, $Du = D_x u$ is the spatial gradient vector of u, and $F = F(x, t) = (f_1(x, t), \cdots, f_n(x, t)) \in L^p(\Omega_T; \mathbb{R}^n)$ is a given vector-valued function.

Here, a vector-valued function

$$\mathbf{a}(\xi, x, t) = (a_1(\xi, x, t), \cdots, a_n(\xi, x, t)) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

is assumed to be a Carathéodory function, namely, measurable in (x, t) and continuous in ξ. We start with the definition that $\mathbf{b}(\xi, x, t)$ is regular, and then we introduce the definition that $\mathbf{a}(\xi, x, t)$ is asymptotically regular as the elliptic case.

Definition 3.1.7. $\mathbf{b}(\xi, x, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is regular if* $\mathbf{b}(\xi, x, t)$ *is a* C^1 *function in* ξ *and there exist positive constants* λ *and* Λ *such that*

$$|\mathbf{b}(\xi, x, t)| + |\xi||D_\xi \mathbf{b}(\xi, x, t)| \leq \Lambda |\xi|^{p-1} \tag{3.15}$$

and

$$D_\xi \mathbf{b}(\xi, x, t)\eta \cdot \eta \geq \lambda |\xi|^{p-2} |\eta|^2 \tag{3.16}$$

for almost every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ *and all* $\xi, \eta \in \mathbb{R}^n$.

Note that the above structural conditions (3.15)-(3.16) imply that there exists a constant $\gamma = \gamma(n, p, \lambda) > 0$ such that for each $\xi, \eta \in \mathbb{R}^n$ and for almost every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$,

$$(\mathbf{b}(\xi, x, t) - \mathbf{b}(\eta, x, t)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^p \quad \text{if} \;\; p \geq 2,$$

$$(\mathbf{b}(\xi, x, t)\mathbf{b}(\eta, x, t)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} \quad \text{if} \;\; 1 < p < 2.$$

Definition 3.1.8. $\mathbf{a}(\xi, x, t)$ *is asymptotically* δ*-regular if there exists a regu-*

lar vector-valued function $\mathbf{b}(\xi, x, t)$ *such that*

$$\limsup_{|\xi|\to\infty} \frac{|\mathbf{a}(\xi, x, t) - \mathbf{b}(\xi, x, t)|}{|\xi|^{p-1}} \leq \delta, \tag{3.17}$$

uniformly with respect to $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

We remark that (3.17) can be reduced to

$$|\mathbf{a}(\xi, x, t) - \mathbf{b}(\xi, x, t)| \leq \omega(|\xi|)(1 + |\xi|^{p-1}) \tag{3.18}$$

for some uniformly bounded function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\limsup_{r\to\infty} \omega(r) \leq \delta$, as the elliptic case.

In order to measure the oscillation of $\mathbf{b}(\xi, x, t)$ over a parabolic cylinder $Q_{(\rho,\theta)}(y, s)$ in the (x, t)-variables, we define

$$\Theta(\mathbf{b}, Q_{(\rho,\theta)}(y, s))(x, t) = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\mathbf{b}(\xi, x, t) - \overline{\mathbf{b}}_{Q_{(\rho,\theta)}(y,s)}(\xi)|}{|\xi|^{p-1}},$$

where $\overline{\mathbf{b}}_{Q_{(\rho,\theta)}(y,s)}(\xi)$ is the integral average of $\mathbf{b}(\xi, \cdot)$ in the (x, t)-variables over $Q_{(\rho,\theta)}(y, s)$ for each fixed $\xi \in \mathbb{R}^n$, as defined by

$$\begin{aligned} \overline{\mathbf{b}}_{Q_{(\rho,\theta)}(y,s)}(\xi) &= \fint_{Q_{(\rho,\theta)}(y,s)} \mathbf{b}(\xi, x, t)\, dxdt \\ &= \frac{1}{|Q_{(\rho,\theta)}(y, s)|} \int_{Q_{(\rho,\theta)}(y,s)} \mathbf{b}(\xi, x, t)\, dxdt. \end{aligned}$$

Definition 3.1.9. $\mathbf{b}(\xi, x, t)$ *is* (δ, R_0)*-vanishing if we have*

$$\sup_{\substack{0<\rho\leq R_0 \\ 0<\theta\leq R_0^2}} \sup_{(y,s)\in\mathbb{R}^n\times\mathbb{R}} \fint_{Q_{(\rho,\theta)}(y,s)} |\Theta(\mathbf{b}, Q_{(\rho,\theta)}(y, s))(x, t)|\, dxdt \leq \delta.$$

Theorem 3.1.10. *For any given* $q \in (1, \infty)$*, assume that* $F \in L^{pq}(\Omega_T; \mathbb{R}^n)$*. Then there exists a constant* $\delta = \delta(n, p, q, \lambda, \Lambda) > 0$ *such that if* $\mathbf{a}(\xi, x, t)$ *is asymptotically* δ*-regular with* $\mathbf{b}(\xi, x, t)$ *which is* (δ, R_0)*-vanishing for some* $R_0 > 0$*, and if* Ω *is* (δ, R_0)*-Reifenberg flat, then a weak solution* u*, which belongs to* $C^0([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$*, of the problem (3.14) satisfies*

$Du \in L^{pq}(\Omega_T;\mathbb{R}^n)$ with the estimate

$$\|Du\|_{L^{pq}(\Omega_T)} \leq c\left(\|F\|_{L^{pq}(\Omega_T)} + 1\right)^d,$$

where $c = c(n,p,q,\lambda,\Lambda,\omega,|\Omega_T|,R_0)$ is a positive constant, and

$$1 \leq d = \begin{cases} \frac{p}{2}, & \text{if} \quad p \geq 2, \\ \frac{2p}{p(n+2)-2n}, & \text{if} \quad \frac{2n}{n+2} < p < 2. \end{cases}$$

Remark 3.1.11. *We point out that the lower bound $\frac{2n}{n+2}$ of p is unavoidable for the type of regularity we are considering here. The exponent d comes from the scaling deficit of the problem (3.14). The presence of the number d is natural, even for the p-Laplacian system, $u_t - \operatorname{div}(|Du|^{p-2}Du) = 0$, see [4].*

3.1.2 Transformation to regular problems

In the following two subsections, for the sake of convenience, we employ the letter c to denote any universal constants which can be explicitly computed in terms of known quantities such as n, p, q, λ, Λ, ω, T and the geometric assumption on Ω, and so c might vary from line to line.

The purpose of this subsection is to transform a given asymptotically regular problem into a suitable regular problem. To do this, of course, we use mainly the asymptotically δ-regular condition on $\mathbf{a}$ and the small BMO assumption on $\mathbf{b}$, $\delta > 0$ being determined.

We start with the elliptic problem. Let $\mathbf{a}(\xi,x)$ be asymptotically δ-regular with $\mathbf{b}(\xi,x)$ which is (δ,R)-vanishing. Then by Definition 3.1.2, we have

$$\limsup_{|\xi|\to\infty} \frac{|\mathbf{a}(\xi,x) - \mathbf{b}(\xi,x)|}{|\xi|^{p-1}} \leq \delta.$$

Now we define $\mathbf{c}(\xi,x)$ by

$$\mathbf{c}(\xi,x) := \frac{\mathbf{a}(\xi,x) - \mathbf{b}(\xi,x)}{|\xi|^{p-1}}, \quad (\xi \neq 0). \tag{3.19}$$

Then clearly $\mathbf{c}(\xi,x)$ is a Carathéodory function. Also there exists $M =$

$M(\delta,\omega) > 1$ such that

$$|\xi| \geq M \quad \Longrightarrow \quad |\mathbf{c}(\xi,x)| \leq 2\delta, \tag{3.20}$$

uniformly with respect to $x \in \mathbb{R}^n$.

We define $\widetilde{\mathbf{c}}(\xi,x)$ by

$$\widetilde{\mathbf{c}}(\xi,x) := \begin{cases} \mathbf{c}(\xi,x) & \text{if} \quad |\xi| \geq M, \\ P[\mathbf{c}(\cdot,x)](\xi) & \text{if} \quad |\xi| < M, \end{cases} \tag{3.21}$$

where

$$P[\mathbf{c}(\cdot,x)](\xi) = \int_{\partial B_M} \mathbf{c}(\eta,x) P(\xi,\eta)\, d\sigma(\eta)$$

is the Poisson integral, and

$$P(\xi,\eta) = \frac{M^2 - |\xi|^2}{M\omega_{n-1}|\xi-\eta|^n}, \quad (\xi \in B_M,\ \eta \in \partial B_M)$$

is the Poisson kernel for the ball $B_M \subset \mathbb{R}^n$. Here, ω_{n-1} is the surface area of the unit sphere ∂B_1 in $\mathbb{R}^n$. Since $P[\mathbf{c}(\cdot,x)](\xi)$ is harmonic in ξ on the ball B_M and $P[\mathbf{c}(\cdot,x)](\xi)$ can be extended continuously in ξ on the closed ball $\overline{B}_M$, we know that $\widetilde{\mathbf{c}}(\xi,x)$ is also a Carathéodory function and that

$$|\widetilde{\mathbf{c}}(\xi,x)| \leq 2\delta, \quad \forall \xi \in \mathbb{R}^n, \tag{3.22}$$

uniformly with respect to $x \in \mathbb{R}^n$, by the maximum principle and (3.20).

Now, for $\xi \neq 0$,

$$\begin{aligned} \mathbf{a}(\xi,x) &= \mathbf{b}(\xi,x) + |\xi|^{p-1}\mathbf{c}(\xi,x) \\ &= \mathbf{b}(\xi,x) + |\xi|^{p-1}\widetilde{\mathbf{c}}(\xi,x) + |\xi|^{p-1}\left(\mathbf{c}(\xi,x) - \widetilde{\mathbf{c}}(\xi,x)\right) \\ &= \mathbf{b}(\xi,x) + |\xi|^{p-1}\widetilde{\mathbf{c}}(\xi,x) + |\xi|^{p-1}\chi_{\{|\xi|<M\}}\left(\mathbf{c}(\xi,x) - \widetilde{\mathbf{c}}(\xi,x)\right), \end{aligned} \tag{3.23}$$

since $\widetilde{\mathbf{c}}(\xi,x) = \mathbf{c}(\xi,x)$ if $|\xi| \geq M$, where $\chi_{\{|\xi|<M\}}$ denotes the characteristic function of the set $\{|\xi| < M\}$.
If, at $\xi = 0$, we define $|\xi|^{p-1}\mathbf{c}(\xi,x)\,|_{\xi=0} := \mathbf{a}(0,x) - \mathbf{b}(0,x)$, then the equation (3.23) holds for all $\xi \in \mathbb{R}^n$.

Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of (3.5). Define

$$\widetilde{\mathbf{b}}(\xi,x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

by

$$\widetilde{\mathbf{b}}(\xi,x) := \mathbf{b}(\xi,x) + |\xi|^{p-1}\widetilde{\mathbf{c}}(Du(x),x). \tag{3.24}$$

Then by (3.23) and (3.24), we have

$$\operatorname{div}\,\mathbf{a}(Du,x) = \operatorname{div}\,\widetilde{\mathbf{b}}(Du,x) + \operatorname{div}\left(|Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x) - \widetilde{\mathbf{c}}(Du,x)\right)\right)$$

in the weak sense. Thus (3.5) implies that $u \in W_0^{1,p}(\Omega)$ is a weak solution of

$$\begin{aligned}\operatorname{div}\,\widetilde{\mathbf{b}}(Du,x) &= \operatorname{div}\,(|F|^{p-2}F) - \operatorname{div}\left(|Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x) - \widetilde{\mathbf{c}}(Du,x)\right)\right)\\ &= \operatorname{div}\left(|F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x) - \mathbf{c}(Du,x)\right)\right)\\ &= \operatorname{div}\,(|G|^{p-2}G) \qquad \text{in } \ \Omega,\end{aligned} \tag{3.25}$$

where G is defined by

$$G = \frac{|F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x) - \mathbf{c}(Du,x)\right)}{\left||F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x) - \mathbf{c}(Du,x)\right)\right|^{\frac{p-2}{p-1}}}, \tag{3.26}$$

if $|F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x) - \mathbf{c}(Du,x)\right) \neq 0$,
and $G = 0$, if $|F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x) - \mathbf{c}(Du,x)\right) = 0$.

Note that

$$|G| = \left||F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x) - \mathbf{c}(Du,x)\right)\right|^{\frac{1}{p-1}}. \tag{3.27}$$

We then have the following lemma.

Lemma 3.1.12. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem (3.5). Let G be given by (3.26). Then $G \in L^p(\Omega;\mathbb{R}^n)$ with the estimate*

$$\|G\|_{L^p(\Omega)} \leq c\left(\|F\|_{L^p(\Omega)} + 1\right), \tag{3.28}$$

where $c = c(p,\omega,|\Omega|,\delta)$ is a positive constant.

Also if $F \in L^{pq}(\Omega;\mathbb{R}^n)$ for some $q \in (1,\infty)$, then $G \in L^{pq}(\Omega;\mathbb{R}^n)$ with the estimate

$$\|G\|_{L^{pq}(\Omega)} \leq c\left(\|F\|_{L^{pq}(\Omega)} + 1\right), \tag{3.29}$$

where $c = c(p,q,\omega,|\Omega|,\delta)$ is a positive constant.

Proof. Note that

$$|\widetilde{\mathbf{c}}(Du,x)| \leq 2\delta < 2, \tag{3.30}$$

uniformly with respect to $x \in \Omega$ by (3.22). Also, by (3.9),

$$\begin{aligned}\left||Du|^{p-1}\mathbf{c}(Du,x)\right| &= |\mathbf{a}(Du,x)-\mathbf{b}(Du,x)| \\ &\le \omega(|Du|)(1+|Du|^{p-1}) \le \|\omega\|_\infty (1+|Du|^{p-1}),\end{aligned}$$

and hence

$$\begin{aligned}\left||Du|^{p-1}\chi_{\{|Du|<M\}}\mathbf{c}(Du,x)\right| &\le \|\omega\|_\infty (1+|Du|^{p-1})\chi_{\{|Du|<M\}} \\ &\le \|\omega\|_\infty (1+M^{p-1}) \le 2\|\omega\|_\infty M^{p-1}. \qquad (3.31)\end{aligned}$$

Therefore by (3.27), (3.30) and (3.31), we have

$$\begin{aligned}|G|^p &= \left||F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)\right|^{\frac{p}{p-1}} \\ &\le c\left(\left(|F|^{p-1}\right)^{\frac{p}{p-1}} + \left(\left(\|\omega\|_\infty+1\right)M^{p-1}\right)^{\frac{p}{p-1}}\right) \\ &= c\left(|F|^p + \left(\|\omega\|_\infty+1\right)^{\frac{p}{p-1}} M^p\right), \qquad (3.32)\end{aligned}$$

where $c = c(p)$ is a positive constant.

Similarly, for $q > 1$, we have

$$|G|^{pq} \le c\left(|F|^{pq} + \left(\|\omega\|_\infty+1\right)^{\frac{pq}{p-1}} M^{pq}\right), \qquad (3.33)$$

where $c = c(p,q)$ is a positive constant.

Therefore, from (3.32), we have $G \in L^p(\Omega;\mathbb{R}^n)$ since $\Omega \subset \mathbb{R}^n$ is a bounded domain and $F \in L^p(\Omega;\mathbb{R}^n)$.

Moreover, we observe from (3.32) that

$$\begin{aligned}\|G\|^p_{L^p(\Omega)} &\le c\left(\|F\|^p_{L^p(\Omega)} + |\Omega|\left(\|\omega\|_\infty+1\right)^{\frac{p}{p-1}} M^p\right) \\ &\le c\left(\|F\|^p_{L^p(\Omega)}+1\right),\end{aligned}$$

or,

$$\|G\|_{L^p(\Omega)} \le c\left(\|F\|_{L^p(\Omega)}+1\right),$$

which is (3.28), where $c = c(p,\omega,|\Omega|,\delta)$ is a positive constant.

Similarly, for $q > 1$, from (3.33), if $F \in L^{pq}(\Omega;\mathbb{R}^n)$, then $G \in L^{pq}(\Omega;\mathbb{R}^n)$

with the estimate

$$\begin{aligned}\|G\|_{L^{pq}(\Omega)}^{pq} &\leq c\left(\|F\|_{L^{pq}(\Omega)}^{pq} + |\Omega|\left(\|\omega\|_\infty + 1\right)^{\frac{pq}{p-1}} M^{pq}\right)\\ &\leq c\left(\|F\|_{L^{pq}(\Omega)}^{pq} + 1\right),\end{aligned}$$

or,

$$\|G\|_{L^{pq}(\Omega)} \leq c\left(\|F\|_{L^{pq}(\Omega)} + 1\right),$$

where $c = c(p, q, \omega, |\Omega|, \delta)$ is a positive constant. □

Lemma 3.1.13. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem (3.52). Assume that $\mathbf{a}(\xi, x)$ is asymptotically δ-regular with $\mathbf{b}(\xi, x)$ which is regular and (δ, R)-vanishing. Then we have*

1. *$\widetilde{\mathbf{b}}(\xi, x)$ is regular, if $0 < \delta < \min\left\{\frac{\lambda}{4(p-1)}, 1\right\}$.*
2. *$\widetilde{\mathbf{b}}(\xi, x)$ is $(5\delta, R)$-vanishing.*

Proof. (1) Since $\mathbf{b}(\xi, x)$ and $|\xi|^{p-1}$ are differentiable in ξ, so is $\widetilde{\mathbf{b}}(\xi, x)$. Let

$$0 < \delta < \min\left\{\frac{\lambda}{4(p-1)}, 1\right\}.$$

Then (3.64) and (3.62) give

$$|\widetilde{\mathbf{b}}(\xi, x)| \leq |\mathbf{b}(\xi, x)| + 2|\xi|^{p-1}. \tag{3.34}$$

Let us note that

$$\begin{aligned}D_\xi \widetilde{\mathbf{b}}(\xi, x) &= D_\xi \mathbf{b}(\xi, x) + \widetilde{\mathbf{c}}(Du(x), x) D_\xi(|\xi|^{p-1})^\top\\ &= D_\xi \mathbf{b}(\xi, x) + \widetilde{\mathbf{c}}(Du(x), x)\left((p-1)|\xi|^{p-3}\xi\right)^\top,\end{aligned} \tag{3.35}$$

and so

$$|D_\xi \widetilde{\mathbf{b}}(\xi, x)| \leq |D_\xi \mathbf{b}(\xi, x)| + 2(p-1)|\xi|^{p-2}. \tag{3.36}$$

We then use (3.34), (3.36) and (3.53) to get that

$$\begin{aligned}|\widetilde{\mathbf{b}}(\xi, x)| + |\xi||D_\xi \widetilde{\mathbf{b}}(\xi, x)| &\leq |\mathbf{b}(\xi, x)| + |\xi||D_\xi \mathbf{b}(\xi, x)| + 2p|\xi|^{p-1}\\ &\leq \Lambda|\xi|^{p-1} + 2p|\xi|^{p-1} = \widetilde{\Lambda}|\xi|^{p-1},\end{aligned} \tag{3.37}$$

where $\widetilde{\Lambda} = \Lambda + 2p$.

On the other hand, by (3.35), (3.54) and (3.62), we conclude that

$$\begin{aligned} D_\xi \widetilde{\mathbf{b}}(\xi, x)\eta \cdot \eta &= D_\xi \mathbf{b}(\xi, x)\eta \cdot \eta + (p-1)|\xi|^{p-3}\widetilde{\mathbf{c}}(Du(x), x)\xi^\top \eta \cdot \eta \\ &\geq \lambda |\xi|^{p-2}|\eta|^2 - 2\delta(p-1)|\xi|^{p-2}|\eta|^2 \\ &= (\lambda - 2(p-1)\delta)\, |\xi|^{p-2}|\eta|^2 \\ &\geq \frac{\lambda}{2}|\xi|^{p-2}|\eta|^2, \end{aligned} \tag{3.38}$$

since $0 < \delta < \dfrac{\lambda}{4(p-1)}$. The assertion (1) now follows from (3.37) and (3.38).

(2) Let $0 < r \leq R$ and $y \in \mathbb{R}^n$. Then for any $\xi \in \mathbb{R}^n \backslash \{0\}$, it follows from (3.64) and (3.62) that

$$\begin{aligned} |\widetilde{\mathbf{b}}(\xi, x) - \overline{\widetilde{\mathbf{b}}}_{B_r(y)}(\xi)| &\leq |\mathbf{b}(\xi, x) - \overline{\mathbf{b}}_{B_r(y)}(\xi)| + 2\delta|\xi|^{p-1} + 2\delta|\xi|^{p-1} \\ &= |\mathbf{b}(\xi, x) - \overline{\mathbf{b}}_{B_r(y)}(\xi)| + 4\delta|\xi|^{p-1}, \end{aligned}$$

and so, we have that

$$\begin{aligned} \Theta(\widetilde{\mathbf{b}}, B_r(y))(x) &= \sup_{\xi \in \mathbb{R}^n \backslash \{0\}} \frac{|\widetilde{\mathbf{b}}(\xi, x) - \overline{\widetilde{\mathbf{b}}}_{B_r(y)}(\xi)|}{|\xi|^{p-1}} \\ &\leq \sup_{\xi \in \mathbb{R}^n \backslash \{0\}} \frac{|\mathbf{b}(\xi, x) - \overline{\mathbf{b}}_{B_r(y)}(\xi)|}{|\xi|^{p-1}} + 4\delta \\ &= \Theta(\mathbf{b}, B_r(y))(x) + 4\delta. \end{aligned}$$

Therefore, since $\mathbf{b}(\xi, x)$ is (δ, R)-vanishing, we conclude that

$$\begin{aligned} &\sup_{0<r\leq R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} |\Theta(\widetilde{\mathbf{b}}, B_r(y))(x)|\; dx \\ &\leq \sup_{0<r\leq R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} |\Theta(\mathbf{b}, B_r(y))(x)|\; dx + 4\delta \\ &\leq \delta + 4\delta = 5\delta, \end{aligned}$$

which implies that $\widetilde{\mathbf{b}}(\xi, x)$ is $(5\delta, R)$-vanishing. This completes the proof. $\square$

In view of Lemma 3.1.13, the asymptotically regular problem (3.5) turns out to be a regular problem. Lemma 3.1.12 and the existing theory for regular problems, see Lemma 3.1.14, are employed to finally have the required estimate (3.13). We return to the next subsection for the complete proof of Theorem 3.1.5.

We next deal with the parabolic problem in a similar way that we have handled for the elliptic problem. For the sake of convenience, we sketch the process of the transformation. Let $\mathbf{a}(\xi, x, t)$ be asymptotically δ-regular with $\mathbf{b}(\xi, x, t)$ which is (δ, R_0)-vanishing for some $R_0 > 0$. Then by Definition 3.1.8, we have

$$\limsup_{|\xi|\to\infty} \frac{|\mathbf{a}(\xi, x, t) - \mathbf{b}(\xi, x, t)|}{|\xi|^{p-1}} \leq \delta.$$

Now we define $\mathbf{c}(\xi, x, t)$ by

$$\mathbf{c}(\xi, x, t) := \frac{\mathbf{a}(\xi, x, t) - \mathbf{b}(\xi, x, t)}{|\xi|^{p-1}}, \quad (\xi \neq 0). \tag{3.39}$$

Then clearly $\mathbf{c}(\xi, x, t)$ is a Carathéodory function and there exists $M = M(\delta, \omega) > 1$ such that

$$|\xi| \geq M \quad \Longrightarrow \quad |\mathbf{c}(\xi, x, t)| \leq 2\delta, \tag{3.40}$$

uniformly with respect to $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.
We define $\widetilde{\mathbf{c}}(\xi, x, t)$ as (3.21). Then $\widetilde{\mathbf{c}}(\xi, x, t)$ is also a Carathéodory function and

$$|\widetilde{\mathbf{c}}(\xi, x, t)| \leq 2\delta, \quad \forall \xi \in \mathbb{R}^n, \tag{3.41}$$

uniformly with respect to $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

We next let $u \in C^0([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ be a weak solution of (3.14). Define

$$\widetilde{\mathbf{b}}(\xi, x, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

by

$$\widetilde{\mathbf{b}}(\xi, x, t) := \mathbf{b}(\xi, x, t) + |\xi|^{p-1} \widetilde{\mathbf{c}}(Du(x, t), x, t). \tag{3.42}$$

Then $\widetilde{\mathbf{b}}(\xi, x, t)$ is regular, provided that $0 < \delta < \min\left\{\frac{\lambda}{4(p-1)}, 1\right\}$ and $\widetilde{\mathbf{b}}(\xi, x, t)$ is $(5\delta, R_0)$-vanishing. Also we have

$$\operatorname{div} \mathbf{a}(Du, x, t) = \operatorname{div} \widetilde{\mathbf{b}}(Du, x, t)$$

$$+\operatorname{div}\left(|Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x,t)-\widetilde{\mathbf{c}}(Du,x,t)\right)\right) \text{ in } \Omega_T,$$

in the weak sense.
Then (3.14) implies that $u \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ is a weak solution of

$$u_t - \operatorname{div}\widetilde{\mathbf{b}}(Du,x,t) = \operatorname{div}\left(|G|^{p-2}G\right) \qquad \text{in } \ \Omega_T, \tag{3.43}$$

where G is defined by

$$G = \frac{|F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x,t)-\widetilde{\mathbf{c}}(Du,x,t)\right)}{\left||F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x,t)-\widetilde{\mathbf{c}}(Du,x,t)\right)\right|^{\frac{p-2}{p-1}}}, \tag{3.44}$$

if $|F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x,t)-\widetilde{\mathbf{c}}(Du,x,t)\right) \neq 0$,
and $G=0$, if $|F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x,t)-\widetilde{\mathbf{c}}(Du,x,t)\right) = 0$.
Note that

$$|G| = \left||F|^{p-2}F + |Du|^{p-1}\chi_{\{|Du|<M\}}\left(\mathbf{c}(Du,x,t)-\widetilde{\mathbf{c}}(Du,x,t)\right)\right|^{\frac{1}{p-1}}.$$

As in the proof of Lemma 3.1.12, we see that $G \in L^p(\Omega_T;\mathbb{R}^n)$ with the estimate

$$\|G\|_{L^p(\Omega_T)} \leq c\left(\|F\|_{L^p(\Omega_T)} + 1\right), \tag{3.45}$$

where $c = c(p,\omega,|\Omega_T|,\delta)$ is a positive constant. We further observe that if $F \in L^{pq}(\Omega_T;\mathbb{R}^n)$ for some $q \in (1,\infty)$, then $G \in L^{pq}(\Omega_T;\mathbb{R}^n)$ with the estimate

$$\|G\|_{L^{pq}(\Omega_T)} \leq c\left(\|F\|_{L^{pq}(\Omega_T)} + 1\right), \tag{3.46}$$

where $c = c(p,q,\omega,|\Omega_T|,\delta)$ is a positive constant.

3.1.3 Proof of Theorems 3.1.5 and 3.1.10

In this subsection we establish the global Calderón-Zygmund theory for asymptotically regular problems. We start with the elliptic problem (3.5). To do this, we first need the following existing theory for regular elliptic problems.

Lemma 3.1.14. *[26] For any given $q \in (1,\infty)$, assume that $F \in L^{pq}(\Omega;\mathbb{R}^n)$. Then there exists a constant $\delta = \delta(n,p,q,\lambda,\Lambda) > 0$ such that if $\mathbf{a}(\xi,x)$ is regular and (δ,R)-vanishing, and if Ω is (δ,R)-Reifenberg flat, then the weak* *solution* $u \in W_0^{1,p}(\Omega)$ *to the problem (3.5) satisfies* $Du \in L^{pq}(\Omega;\mathbb{R}^n)$ *with the estimate*

$$\|Du\|_{L^{pq}(\Omega)} \leq c\,\|F\|_{L^{pq}(\Omega)}\,,$$

where $c = c(n,p,q,\lambda,\Lambda,|\Omega|)$ *is a positive constant.*

We are now ready to prove Theorem 3.1.5.

Proof of Theorem 3.1.5. Let $\delta_0 > 0$ be the universal constant which is given in Lemma 3.1.14, and let

$$\delta_1 := \min\left\{\frac{\lambda}{4(p-1)},\ 1\right\}.$$

We set $\delta = \frac{1}{5}\min\{\delta_0,\delta_1\} > 0$.

Now let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem (3.5). Assume that $\mathbf{a}(\xi,x)$ is asymptotically δ-regular with $\mathbf{b}(\xi,x)$ which is regular and (δ,R)-vanishing. Then by Lemma 3.1.13, $\widetilde{\mathbf{b}}(\xi,x)$ is regular and $(5\delta,R)$-vanishing, and so (δ_0,R)-vanishing. According to Lemma 3.1.12, we have $G \in L^{pq}(\Omega;\mathbb{R}^n)$. We then apply Lemma 3.1.14 to $G \in L^{pq}(\Omega;\mathbb{R}^n)$ and $\widetilde{\mathbf{b}}(\xi,x)$ to discover $Du \in L^{pq}(\Omega;\mathbb{R}^n)$ with the estimate

$$\|Du\|_{L^{pq}(\Omega)} \leq c\,\|G\|_{L^{pq}(\Omega)}\,, \tag{3.47}$$

where $c = c(n,p,q,\lambda,\Lambda,|\Omega|)$ is a positive constant.

But then Lemma 3.1.12 and (3.47) imply

$$\|Du\|_{L^{pq}(\Omega)} \leq c\left(\|F\|_{L^{pq}(\Omega)} + 1\right), \tag{3.48}$$

for some positive constant $c = c(n,p,q,\lambda,\Lambda,\omega,|\Omega|)$. □

We next return to the parabolic problem (3.14). As the elliptic case, we need the following existing theory for regular parabolic problems.

Lemma 3.1.15. *[23] For any given* $q \in (1,\infty)$*, assume that* $F \in L^{pq}(\Omega_T;\mathbb{R}^n)$*. Then there exists a constant* $\delta = \delta(n,p,q,\lambda,\Lambda) > 0$ *such that if* $\mathbf{a}(\xi,x,t)$ *is regular and* (δ,R_0)*-vanishing for some* $R_0 > 0$*, and if* Ω *is* (δ,R_0)*-Reifenberg flat, then the weak solution* $u \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ *to the* *problem (3.14) satisfies $Du \in L^{pq}(\Omega_T;\mathbb{R}^n)$ with the estimate*

$$\|Du\|_{L^{pq}(\Omega_T)} \leq c\left(\|F\|_{L^{pq}(\Omega_T)}+1\right)^d,$$

where $c = c(n,p,q,\lambda,\Lambda,|\Omega_T|,R_0)$ is a positive constant, and

$$1 \leq d = \begin{cases} \frac{p}{2}, & \text{if} \quad p \geq 2, \\ \frac{2p}{p(n+2)-2n}, & \text{if} \quad \frac{2n}{n+2} < p < 2. \end{cases}$$

Now we prove Theorem 3.1.10.

Proof of Theorem 3.1.10. Let $\delta_0 > 0$ be the universal constant which is given in Lemma 3.1.15. Let

$$\delta_1 = \min\left\{\frac{\lambda}{4(p-1)},\ 1\right\},$$

and set $\delta = \frac{1}{5}\min\{\delta_0,\delta_1\} > 0$ as in the proof of Theorem 3.1.5.

Now let $u \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ be a weak solution of the problem (3.14). Assume that $\mathbf{a}(\xi,x,t)$ is asymptotically δ-regular with $\mathbf{b}(\xi,x,t)$ which is regular and (δ,R_0)-vanishing. Then $\widetilde{\mathbf{b}}(\xi,x,t)$ is regular and $(5\delta,R_0)$-vanishing, and so (δ_0,R_0)-vanishing. From (3.46), we get $G \in L^{pq}(\Omega_T;\mathbb{R}^n)$. We then apply Lemma 3.1.15 to $G \in L^{pq}(\Omega_T;\mathbb{R}^n)$ and $\widetilde{\mathbf{b}}(\xi,x,t)$ to discover $Du \in L^{pq}(\Omega_T;\mathbb{R}^n)$ with the estimate

$$\|Du\|_{L^{pq}(\Omega_T)} \leq c\left(\|G\|_{L^{pq}(\Omega_T)}+1\right)^d, \tag{3.49}$$

where $c = c(n,p,q,\lambda,\Lambda,|\Omega_T|,R_0)$ is a positive constant.

Then (3.46) and (3.49) imply

$$\|Du\|_{L^{pq}(\Omega_T)} \leq c\left(\|F\|_{L^{pq}(\Omega_T)}+1\right)^d, \tag{3.50}$$

for some positive constant $c = c(n,p,q,\lambda,\Lambda,\omega,|\Omega_T|,R_0)$. □

3.2 $W^{1,q}$-estimates for nonlinear elliptic obstacle problems with asymptotically regular nonlinearities

In this section, we are concerned with a nonlinear elliptic problem with an irregular obstacle in a bounded nonsmooth domain. We want to establish a global Calderón-Zygmund estimate for an elliptic obstacle problem of p-Laplacian type by proving that the gradient of a weak solution is as globally integrable as both the gradient of the obstacle function and the nonhomogeneous term, provided that the nonlinearity has a sufficient asymptotic regularity, the associated nonlinearity has a small BMO, and the boundary of the domain is sufficiently flat in the Reifenberg sense.

This work is a natural outgrowth of the recent work in [20] where the global Calderón-Zygmund estimate for a nonlinear elliptic equation of p-Laplacian type was obtained. Here we deal with discontinuous obstacles, and want to provide an optimal global Calderón-Zygmund theory for elliptic variational inequalities. To do this, we properly adopt the approach used in [20], in order to get the required estimate for the irregular obstacle problem based on a suitable transformation of the nonlinear operator in the variational form.

3.2.1 Hypotheses and main results

Let Ω be a bounded open domain in $\mathbb{R}^n$ with $n \geq 2$ and let $1 < p < \infty$ be a fixed real number. For an obstacle function $\psi \in W^{1,p}(\Omega)$ with $\psi \leq 0$ a.e. on $\partial\Omega$, we consider an convex admissible set

$$\mathcal{A} = \{\phi \in W_0^{1,p}(\Omega) : \phi \geq \psi \quad \text{a.e. in } \Omega\}. \tag{3.51}$$

and a function $u \in \mathcal{A}$ satisfying the following variational inequality

$$\int_\Omega \mathbf{a}(Du, x) \cdot D(\phi - u)\, dx \geq \int_\Omega |F|^{p-2} F \cdot D(\phi - u)\, dx \text{ for all } \phi \in \mathcal{A}, \tag{3.52}$$

where $F \in L^p(\Omega; \mathbb{R}^n)$ is given, as is the Carathéodory vector-valued function

$$\mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

namely, it is continuous in ξ and measurable in x. We then call such a function $u \in \mathcal{A}$ to be a weak solution to the variational inequality (3.52).

In this section we are mainly interested in the case that $\mathbf{a}(\xi, x)$ is asymptotically regular. This is the case that it is getting closer to some regular function $\mathbf{b}(\xi, x)$ as $|\xi|$ goes to infinity.

Definition 3.2.1. *We say that a Carathéodory function $\mathbf{b}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is regular, if there exist positive constants λ and Λ such that*

$$|\mathbf{b}(\xi, x)| + |\xi||D_\xi \mathbf{b}(\xi, x)| \leq \Lambda |\xi|^{p-1} \tag{3.53}$$

and

$$D_\xi \mathbf{b}(\xi, x)\eta \cdot \eta \geq \lambda |\xi|^{p-2}|\eta|^2 \tag{3.54}$$

for almost every $x \in \mathbb{R}^n$ and all $\xi, \eta \in \mathbb{R}^n$.

Note that the above structural conditions (3.53)-(3.54) imply the following monotonicity conditions: for any $\xi,\ \eta \in \mathbb{R}^n$ and for almost every $x \in \mathbb{R}^n$,

$$(\mathbf{b}(\xi, x) - \mathbf{b}(\eta, x)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^p \quad \text{ if } \ p \geq 2,$$

$$(\mathbf{b}(\xi, x) - \mathbf{b}(\eta, x)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} \quad \text{ if } \ 1 < p < 2,$$

where γ is a positive constant depending only on n, p, and λ.

We are now in a position to introduce the concept of the asymptotically regular condition on $\mathbf{a}(\xi, x)$.

Definition 3.2.2. *Let $\mathbf{b}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be regular. Then we say that $\mathbf{a}(\xi, x)$ is asymptotically δ-regular with $\mathbf{b}(\xi, x)$ if there exists a uniformly bounded nonnegative function ω on $[0, \infty)$ such that*

$$\limsup_{\rho \to \infty} \omega(\rho) \leq \delta \tag{3.55}$$

and

$$|\mathbf{a}(\xi, x) - \mathbf{b}(\xi, x)| \leq \omega(|\xi|)(1 + |\xi|^{p-1}) \tag{3.56}$$

for almost every $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$.

Remark 3.2.3. *If $\mathbf{a}(\xi, x)$ is asymptotically δ-regular with $\mathbf{b}(\xi, x)$, then we see that*

$$\limsup_{|\xi| \to \infty} \sup_{x \in \mathbb{R}^n} \frac{|\mathbf{a}(\xi, x) - \mathbf{b}(\xi, x)|}{|\xi|^{p-1}} \leq 2\delta, \tag{3.57}$$

which means that for a sufficiently small $\delta > 0$, $\mathbf{a}(\xi, x)$ is in a regular range near infinity for $|\xi|$.

Throughout this section we always assume that $\mathbf{b}(\xi, x)$ is regular and $\mathbf{a}(\xi, x)$ is asymptotically δ-regular with $\mathbf{b}(\xi, x)$, where δ is to be determined later.

In this work we study an optimal global Calderón-Zygmund estimate for the asymptotically regular obstacle problem (3.52). To do this, we need to employ an existing result for the following variational inequality when $\mathbf{a}(\xi, x)$ is replaced by $\mathbf{b}(\xi, x)$ in (3.52):

$$\int_\Omega \mathbf{b}(Du, x) \cdot D(\phi - u)\, dx \geq \int_\Omega |F|^{p-2} F \cdot D(\phi - u)\, dx \text{ for all } \phi \in \mathcal{A}. \tag{3.58}$$

As usual, we need to add a minimal regularity condition on the nonlinearity to the structure conditions (3.53)-(3.54), and a lower level of geometric condition on the boundary of the domain, as we now introduce.

Definition 3.2.4. *We say that $\mathbf{b}(\xi, x)$ is (δ, R)-vanishing, if we have*

$$\sup_{0 < r \leq R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} |\Theta(\mathbf{b}, B_r(y))(x)|\, dx \leq \delta,$$

where

$$\Theta(\mathbf{b}, B_r(y))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{b}(\xi, x) - \overline{\mathbf{b}}_{B_r(y)}(\xi)|}{|\xi|^{p-1}}$$

for the integral average $\overline{\mathbf{b}}_{B_r(y)}(\xi)$ of $\mathbf{b}(\xi, \cdot)$ in the variable x over $B_r(y)$.

Definition 3.2.5. *We say that Ω is a (δ, R)-Reifenberg flat domain if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{z_1, \cdots, z_n\}$ with the center x, which can depend on x and r, such that*

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}.$$

Remark 3.2.6. *On the above definitions, R can be any positive number from a scaling invariance property of the problem (3.58), while δ is invariant under such a scaling. The domain Ω is a δ-Reifenberg flat one whose boundary is very irregular and can go beyond the Lipschitz category. We refer to [85, 86, 108, 118] and the references therein for a further discussion on Reifenberg flat domains.*

We need the following global Calderón-Zygmund estimate for the regular obstacle problem (3.58).

Lemma 3.2.7. *[14] For any given $q \in (1,\infty)$, we assume that $D\psi \in L^{pq}(\Omega;\mathbb{R}^n)$ and $F \in L^{pq}(\Omega;\mathbb{R}^n)$. Then there is a constant $\delta = \delta(n,p,q,\lambda,\Lambda) > 0$ such that if Ω is (δ,R)-Reifenberg flat and $\mathbf{b}(\xi,x)$ is (δ,R)-vanishing, then a weak solution $u \in \mathcal{A}$ to the variation inequality (3.58) satisfies $Du \in L^{pq}(\Omega;\mathbb{R}^n)$ with the estimate*

$$\|Du\|_{L^{pq}(\Omega)} \leq c\left(\|D\psi\|_{L^{pq}(\Omega)} + \|F\|_{L^{pq}(\Omega)}\right),$$

where c is a positive constant depending only on n, p, q, λ, Λ and $|\Omega|$.

We now state the main result in this work for the asymptotically regular obstacle problem (3.52).

Theorem 3.2.8. *Suppose that $D\psi \in L^{pq}(\Omega;\mathbb{R}^n)$ and $F \in L^{pq}(\Omega;\mathbb{R}^n)$ for any $q \in (1,\infty)$. Then there exists a small constant $\delta = \delta(n,p,q,\lambda,\Lambda) > 0$ such that if Ω is (δ,R)-Reifenberg flat and $\mathbf{a}(\xi,x)$ is asymptotically δ-regular with $\mathbf{b}(\xi,x)$ being (δ,R)-vanishing, then a weak solution $u \in \mathcal{A}$ to the variation inequality (3.52) satisfies $Du \in L^{pq}(\Omega;\mathbb{R}^n)$ with the estimate*

$$\|Du\|_{L^{pq}(\Omega)} \leq c\left(\|D\psi\|_{L^{pq}(\Omega)} + \|F\|_{L^{pq}(\Omega)} + 1\right),$$

where c is a positive constant depending only on n, p, q, λ, Λ, ω and $|\Omega|$.

3.2.2 Proof of Theorem 3.2.8

In this subsection we prove our main result, Theorem 3.2.8. The main idea is to construct a regular problem using the Poisson's formula.

We first define a vector-valued function $\mathbf{c}(\xi,x) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$|\xi|^{p-1}\mathbf{c}(\xi,x) = \mathbf{a}(\xi,x) - \mathbf{b}(\xi,x). \tag{3.59}$$

Then (3.57) implies that there exists $M_0 > 1$ such that

$$|\xi| \geq M_0 \implies |\mathbf{c}(\xi,x)| \leq 2\delta, \tag{3.60}$$

uniformly in $x \in \mathbb{R}^n$.

For any fixed $x \in \mathbb{R}^n$, we consider the Poisson integral

$$P[\mathbf{c}(\cdot, x)](\xi) := \int_{\partial B_{M_0}} \mathbf{c}(\eta, x) K(\xi, \eta) \, d\sigma(\eta) \text{ for } \xi \in B_{M_0},$$

where

$$K(\xi, \eta) = \frac{M_0^2 - |\xi|^2}{M_0 \omega_{n-1} |\xi - \eta|^n}, \quad (\xi \in B_{M_0}, \ \eta \in \partial B_{M_0})$$

is the Poisson kernel for the ball $B_{M_0} \subset \mathbb{R}^n$ with the radius M_0, where ω_{n-1} is the surface area of the unit sphere ∂B_1 in $\mathbb{R}^n$. We next another vector-valued function $\widetilde{\mathbf{c}}(\xi, x)$ by

$$\widetilde{\mathbf{c}}(\xi, x) := \begin{cases} \mathbf{c}(\xi, x) & \text{if} \quad |\xi| \geq M_0, \\ P[\mathbf{c}(\cdot, x)](\xi) & \text{if} \quad |\xi| < M_0. \end{cases} \tag{3.61}$$

Then $\widetilde{\mathbf{c}}(\xi, x)$ is a Carathéodory function. According to (3.60) and the maximum principle, we see that

$$|\widetilde{\mathbf{c}}(\xi, x)| \leq 2\delta, \quad \forall \xi \in \mathbb{R}^n, \tag{3.62}$$

uniformly in $x \in \mathbb{R}^n$.

We observe that

$$\begin{aligned} \mathbf{a}(\xi, x) &= \mathbf{b}(\xi, x) + |\xi|^{p-1} \mathbf{c}(\xi, x) \\ &= \mathbf{b}(\xi, x) + |\xi|^{p-1} \widetilde{\mathbf{c}}(\xi, x) + |\xi|^{p-1} \chi_{\{|\xi| < M_0\}} \left(\mathbf{c}(\xi, x) - \widetilde{\mathbf{c}}(\xi, x) \right), \end{aligned} \tag{3.63}$$

where $\chi_{\{|\xi|<M_0\}}$ is the characteristic function on the set $\{\xi \in \mathbb{R}^n : |\xi| < M_0\}$.

Here we introduce our main tool, which is to transfer the asymptotically regular obstacle problem into a regular one. For a given weak solution $u \in \mathcal{A}$ of (3.52), we define

$$\widetilde{\mathbf{b}}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

by

$$\widetilde{\mathbf{b}}(\xi, x) := \mathbf{b}(\xi, x) + |\xi|^{p-1} \widetilde{\mathbf{c}}(Du(x), x). \tag{3.64}$$

We are now ready to prove our main result.

Proof of Theorem 3.2.8. From (3.63)-(3.64), we have

$$\mathbf{a}(Du, x) = \widetilde{\mathbf{b}}(Du, x) + |Du|^{p-1} \chi_{\{|Du|<M_0\}} \left(\mathbf{c}(Du, x) - \widetilde{\mathbf{c}}(Du, x) \right),$$

Then (3.52) implies that

$$\begin{aligned}&\int_\Omega \widetilde{\mathbf{b}}(Du,x)\cdot D(\phi-u)\,dx\\ &\geq \int_\Omega \left\{|F|^{p-2}F+|Du|^{p-1}\chi_{\{|Du|<M_0\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)\right\}\cdot D(\phi-u)\,dx\end{aligned} \tag{3.65}$$

for all $\phi\in\mathcal{A}$. Note from (3.62) that

$$|\widetilde{\mathbf{c}}(Du,x)|\leq 2\delta<2 \text{ for all } x\in\Omega. \tag{3.66}$$

We also note from (3.56) that

$$\begin{aligned}\left||Du|^{p-1}\chi_{\{|Du|<M_0\}}\mathbf{c}(Du,x)\right| &= |\mathbf{a}(Du,x)-\mathbf{b}(Du,x)|\chi_{\{|Du|<M_0\}}\\ &\leq \|\omega\|_\infty\,(1+|Du|^{p-1})\chi_{\{|Du|<M_0\}}\\ &\leq \|\omega\|_\infty\,(1+M_0^{p-1})\\ &\leq 2\,\|\omega\|_\infty\,M_0^{p-1}.\end{aligned} \tag{3.67}$$

Hence from (3.66) and (3.67), we conclude that the integrand of the right hand side in (3.65) is well defined.

We recall that u is a weak solution to the following variational inequality

$$\int_\Omega \widetilde{\mathbf{b}}(Du,x)\cdot D(\phi-u)\,dx \geq \int_\Omega |G|^{p-2}G\cdot D(\phi-u)\,dx \text{ for all } \phi\in\mathcal{A}, \tag{3.68}$$

where G is defined by

$$G=\frac{|F|^{p-2}F+|Du|^{p-1}\chi_{\{|Du|<M_0\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)}{\left||F|^{p-2}F+|Du|^{p-1}\chi_{\{|Du|<M_0\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)\right|^{\frac{p-2}{p-1}}}, \tag{3.69}$$

if

$$\left||F|^{p-2}F+|Du|^{p-1}\chi_{\{|Du|<M_0\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)\right|\neq 0,$$

while $G=0$, if

$$\left||F|^{p-2}F+|Du|^{p-1}\chi_{\{|Du|<M_0\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)\right|=0.$$

Then it is clear that G belongs to $L^{pq}(\Omega;\mathbb{R}^n)$ with the estimate

$$\begin{aligned}\|G\|_{L^{pq}(\Omega)} &\le c\left(\|F\|_{L^{pq}(\Omega)} + |\Omega|^{\frac{1}{pq}}\left(1+\|\omega\|_\infty\right)^{\frac{1}{p-1}} M_0\right)\\ &\le c\left(\|F\|_{L^{pq}(\Omega)} + 1\right),\end{aligned} \tag{3.70}$$

where $c = c(n,\delta,p,q,\omega,|\Omega|)$ is a positive constant.

Recalling Lemma 3.1.13 and using (3.68), (3.69) and (3.70), we employ Lemma 3.2.7 with $\mathbf{b}(\xi,x)$ replaced by $\widetilde{\mathbf{b}}(\xi,x)$ and F replaced by G, respectively, to complete the proof. □

3.3 $W^{1,q(\cdot)}$-estimates for asymptotically regular problems of $p(x)$-Laplacian type

In this section, we consider the following nonhomogenous asymptotically regular problem of $p(x)$-Laplacian type

$$\begin{cases} \operatorname{div} \mathbf{a}(Du,x) = \operatorname{div}\left(|F|^{p(x)-2}F\right) & \text{in} \quad \Omega,\\ u = 0 & \text{on} \quad \partial\Omega,\end{cases} \tag{3.71}$$

with a discontinuous nonlinearity $\mathbf{a}$ and a given F. Here, Ω is a nonsmooth bounded domain in $\mathbb{R}^n$ with $n \ge 2$, and $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ is a continuous function satisfying

$$1 < \gamma_1 \le p(x) \le \gamma_2 < \infty \tag{3.72}$$

for some constants γ_1 and γ_2. The aim of this section is to establish a global Calderón-Zygmund estimate in the setting of variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ under possibly optimal conditions on the nonlinearity $\mathbf{a}$ and the boundary of Ω by essentially proving that

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \implies |Du|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \tag{3.73}$$

holds true for $q(\cdot) : \Omega \to (1,\infty)$ satisfying

$$1 < \gamma_3 \le q(x) \le \gamma_4 < \infty \tag{3.74}$$

for some constants γ_3 and γ_4.

3.3.1 Hypotheses and main results

Let Ω be a bounded open domain in $\mathbb{R}^n$ with $n \geq 2$ and let $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ be a given continuous function satisfying (3.72). We then consider the following nonlinear elliptic problem of $p(x)$-Laplacian type in divergence form:

$$\left\{ \begin{array}{rcll} \text{div } \mathbf{a}(Du, x) & = & \text{div } (|F|^{p(x)-2}F) & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{3.75}$$

where $F = (f^1, \cdots, f^n) \in L^{p(\cdot)}(\Omega; \mathbb{R}^n)$ is the nonhomogeneous term.

Here, a vector-valued function

$$\mathbf{a}(\xi, x) = \big(a^1(\xi, x), \cdots, a^n(\xi, x)\big) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is assumed to be a Carathéodory function, namely, measurable in x and continuous in ξ. To introduce the definition that $\mathbf{a}(\xi, x)$ is asymptotically regular, we start with the definition that a vector-valued function $\mathbf{b}(\xi, x)$ is regular.

Definition 3.3.1. *We say that a Carathéodory function* $\mathbf{b}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is regular with $p(x)$-growth if there exist* $0 < \lambda \leq \Lambda < \infty$ *and* $0 \leq \mu \leq 1$ *such that*

$$|\mathbf{b}(\xi, x)| + (\mu^2 + |\xi|^2)^{\frac{1}{2}} |D_\xi \mathbf{b}(\xi, x)| \leq \Lambda (\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}} \tag{3.76}$$

and

$$D_\xi \mathbf{b}(\xi, x)\eta \cdot \eta \geq \lambda (\mu^2 + |\xi|^2)^{\frac{p(x)-2}{2}} |\eta|^2 \tag{3.77}$$

for almost every $x \in \mathbb{R}^n$ *and all* $\xi, \eta \in \mathbb{R}^n$.

Here, $D_\xi \mathbf{b}(\xi, x)$ denotes the Jacobian matrix of $\mathbf{b}(\xi, x)$ with respect to the variable ξ. We remark that the above structural conditions (3.76)-(3.77) imply the following monotonicity condition: for each $\xi, \eta \in \mathbb{R}^n$ and for almost every $x \in \mathbb{R}^n$,

$$(\mathbf{b}(\xi, x) - \mathbf{b}(\eta, x)) \cdot (\xi - \eta) \geq \nu |\xi - \eta|^2 (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p(x)-2}{2}},$$

where ν is a positive constant depending only on n, γ_1, γ_2 and λ. In particular, for the case $p(x) \geq 2$, it can be reduced to

$$(\mathbf{b}(\xi, x) - \mathbf{b}(\eta, x)) \cdot (\xi - \eta) \geq \nu |\xi - \eta|^{p(x)}.$$

In order to measure the oscillation of $\frac{\mathbf{b}(\xi,x)}{(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}}$ over a ball $B_r(y)$ in the variable x, we define

$$\Theta(\mathbf{b}, B_r(y))(x) := \sup_{\xi\in\mathbb{R}^n} \left| \frac{\mathbf{b}(\xi,x)}{(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}} - \overline{\left(\frac{\mathbf{b}(\xi,\cdot)}{(\mu^2+|\xi|^2)^{\frac{p(\cdot)-1}{2}}}\right)}_{B_r(y)} \right|$$

(if $\mu = 0$, the above supremum runs over all $\xi \in \mathbb{R}^n\backslash\{0\}$). We remark that if $\mathbf{b}(\xi,x)$ is a regular function with $p(x)$-growth satisfying (3.76) and (3.77), then we have $|\Theta(\mathbf{b}, B_r(y))(x)| \le 2\Lambda$ for all $x \in B_r(y)$.

Throughout this section, $0 < \delta < \frac{1}{8}$ is a universal constant to be defined later so that our main result holds.

Definition 3.3.2. $\mathbf{b}(\xi,x)$ *is* (δ, R)*-vanishing if*

$$\sup_{0<r\le R}\ \sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} |\Theta(\mathbf{b}, B_r(y))(x)|\ dx \le \delta. \tag{3.78}$$

We next state the geometric assumption on the boundary of the domain.

Definition 3.3.3. Ω *is* (δ, R)*-Reifenberg flat if for every* $x \in \partial\Omega$ *and every* $r \in (0, R]$*, there exists a coordinate system* $\{y_1, \cdots, y_n\}$*, which can depend on* r *and* x *so that* $x = 0$ *in this coordinate system and that*

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}.$$

Note that a Reifenberg flat domain can go beyond Lipschitz category, not necessarily given by graphs. The boundary of a Reifenberg flat domain is so rough that even the unit normal vector can not be well defined there in the usual sense. However its boundary is well approximated by hyperplanes at every point and at every scale. We refer to [70, 108, 118] and references therein regarding its applications in analysis and geometry.

In this section, we are mainly concerned with a global Calderón-Zygmund type estimate for the asymptotically regular problem with $p(x)$-growth which we now introduce.

Definition 3.3.4. *Let* $\mathbf{b}(\xi,x)$ *be a regular function with* $p(x)$*-growth as in Definition 3.3.1. Then we say that* $\mathbf{a}(\xi,x)$ *is asymptotically* δ*-regular with* $\mathbf{b}(\xi,x)$ *if there exists a uniformly bounded nonnegative function* $\varphi : [0,\infty) \to$

$[0, \infty)$ *such that*

$$\limsup_{r\to\infty} \varphi(r) \leq \delta \tag{3.79}$$

and

$$|\mathbf{a}(\xi, x) - \mathbf{b}(\xi, x)| \leq \varphi(|\xi|)(1 + |\xi|^{p(x)-1}) \tag{3.80}$$

for almost every $x \in \mathbb{R}^n$ *and all* $\xi \in \mathbb{R}^n$.

Remark 3.3.5. *If* $\mathbf{a}(\xi, x)$ *is asymptotically* δ*-regular with* $\mathbf{b}(\xi, x)$*, then we see that*

$$\limsup_{|\xi|\to\infty} \frac{|\mathbf{a}(\xi, x) - \mathbf{b}(\xi, x)|}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} \leq \delta,$$

uniformly with respect to $x \in \mathbb{R}^n$.

We point out that the notion of an asymptotically regular function in Definition 3.3.4 is weaker than that of an asymptotically regular function, which was considered in the previous papers [10, 62, 82, 112], as we do not require that $\lim_{r\to\infty} \varphi(r) = 0$. Indeed, we can include some oscillating cases, for example,

$$\mathbf{a}(\xi, x) = (\mu^2 + |\xi|^2)^{\frac{p(x)-2}{2}} \xi + \delta \sin(|\xi|^2)(\mu^2 + |\xi|^2)^{\frac{p(x)-2}{2}} \xi$$

or

$$\mathbf{a}(\xi, x) = (\mu^2 + |\xi|^2)^{\frac{p(x)-2}{2}} \xi + \delta h(x)(\mu^2 + |\xi|^2)^{\frac{p(x)-2}{2}} \xi,$$

where $h : \mathbb{R}^n \to [-1, 1]$ is a measurable function and $\delta > 0$ is a constant.

To prove (3.73), we assume that $q(\cdot)$ satisfies (3.74) and that $q(\cdot)$ is log-Hölder continuous, that is, $q(\cdot)$ admits a modulus of continuity $\rho : [0, \infty) \to [0, \infty)$ satisfying

$$\sup_{0<r\leq 1} \rho(r) \log\left(\frac{1}{r}\right) \leq L \tag{3.81}$$

for some constant $L > 0$. In addition, we shall assume that $p(\cdot)$ admits a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ satisfying

$$\sup_{0<r\leq R} \omega(r) \log\left(\frac{1}{r}\right) \leq \delta. \tag{3.82}$$

For the sake of simplicity, we shall denote

$$\texttt{data} \equiv \texttt{data}(n, \lambda, \Lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L).$$

The following lemma shows that a global Calderón-Zygmund type estimate holds for a regular problem with $p(x)$-growth.

Lemma 3.3.6. *[22] Suppose that $\mathbf{a}(\xi,x)$ is regular with $p(x)$-growth. Then there exists a unique solution $v \in W_0^{1,p(\cdot)}(\Omega)$ to the problem (3.75). Moreover, there exists a constant $\delta = \delta(\mathtt{data}) > 0$ such that if $\mathbf{a}(\xi,x)$ is (δ,R)-vanishing and Ω is (δ,R)-Reifenberg flat for some $R>0$, then the Calderón-Zygmund type relation*

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \implies |Dv|^{p(\cdot)} \in L^{q(\cdot)}(\Omega)$$

holds with the estimate

$$\int_\Omega |Dv|^{p(x)q(x)} dx \leq c \left(\int_\Omega |F|^{p(x)q(x)} dx + 1 \right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}},$$

where $c>0$ is a constant depending on $\mathtt{data}$, $\omega(\cdot)$, $\rho(\cdot)$, R and Ω.

We now state the main result for an asymptotically regular problem with $p(x)$-growth.

Theorem 3.3.7. *Let $u \in W_0^{1,p(\cdot)}(\Omega)$ be a weak solution to the problem (3.75). There exists a constant $\delta = \delta(\mathtt{data}) > 0$ such that if $\mathbf{a}(\xi,x)$ is asymptotically δ-regular with $\mathbf{b}(\xi,x)$ satisfying (3.76), (3.77) and the (δ,R)-vanishing condition (3.78), and if Ω is (δ,R)-Reifenberg flat for some $R>0$, then the Calderón-Zygmund type relation*

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \implies |Du|^{p(\cdot)} \in L^{q(\cdot)}(\Omega)$$

holds with the estimate

$$\int_\Omega |Du|^{p(x)q(x)} dx \leq c \left(\int_\Omega |F|^{p(x)q(x)} dx + 1 \right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}},$$

where $c>0$ is a constant depending on $\mathtt{data}$, $\varphi(\cdot)$, $\omega(\cdot)$, $\rho(\cdot)$, R and Ω.

Remark 3.3.8. *As a consequence of Theorem 3.3.7, if $q(\cdot) \geq \gamma_3 > \frac{n}{\gamma_1}$, then it follows directly from the Sobolev-Morrey embedding that u is Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{n}{\gamma_1\gamma_3} > 0$.*

3.3.2 Proof of Theorem 3.3.7

In this subsection, for the sake of convenience, we employ the letter c to denote any universal constants which can be explicitly computed in terms of known quantities such as $\mathtt{data}$, R, $\varphi(\cdot)$, $\omega(\cdot)$, $\rho(\cdot)$ and the geometric assumption on Ω, and so c might vary from line to line.

The purpose of this subsection is twofold. One is to find a suitable regular problem from a given asymptotically regular problem. The other is, by means of a method of transformation, to finally establish the global Calderón-Zygmund type estimate for the asymptotically regular problem. In the process, of course, we use mainly the asymptotically δ-regular condition on $\mathbf{a}$ and the small BMO assumption on $\mathbf{b}$.

Let $\mathbf{a}(\xi, x)$ be asymptotically δ-regular with $\mathbf{b}(\xi, x)$ which is a regular function with $p(x)$-growth. Then by Definition 3.3.4, we have

$$\limsup_{|\xi|\to\infty} \frac{|\mathbf{a}(\xi, x) - \mathbf{b}(\xi, x)|}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} \leq \delta,$$

uniformly with respect to $x \in \mathbb{R}^n$. Now we define $\mathbf{c}(\xi, x)$ by

$$\mathbf{c}(\xi, x) := \frac{\mathbf{a}(\xi, x) - \mathbf{b}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}}, \quad (\xi, x \in \mathbb{R}^n). \tag{3.83}$$

Then clearly $\mathbf{c}(\xi, x)$ is a Carathéodory function. Also note that there exists $M > 1$ such that

$$|\mathbf{c}(\xi, x)| \leq 2\delta, \quad \forall \xi, x \in \mathbb{R}^n \text{ with } |\xi| \geq M. \tag{3.84}$$

We now consider the following Poisson kernel for the ball $B_M \subset \mathbb{R}^n$:

$$P(\xi, \eta) = \frac{M^2 - |\xi|^2}{\alpha_{n-1} M |\xi - \eta|^n}, \quad (\xi \in B_M,\ \eta \in \partial B_M),$$

where α_{n-1} is the surface area of the unit sphere $\partial B_1 \subset \mathbb{R}^n$. For each fixed $x \in \mathbb{R}^n$, we next define $\widetilde{\mathbf{c}}(\cdot, x) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\widetilde{\mathbf{c}}(\xi, x) := \begin{cases} \mathbf{c}(\xi, x) & \text{if} \quad |\xi| \geq M, \\ P[\mathbf{c}(\cdot, x)](\xi) & \text{if} \quad |\xi| < M, \end{cases} \tag{3.85}$$

where

$$P[\mathbf{c}(\cdot,x)](\xi) = \int_{\partial B_M} \mathbf{c}(\eta,x)P(\xi,\eta)\, d\sigma(\eta)$$

is the Poisson integral of the vector-valued function $\mathbf{c}(\cdot,x)$. Then $\widetilde{\mathbf{c}}(\xi,x)$ is harmonic in ξ on the ball B_M and it is a Carathéodory function since $P[\mathbf{c}(\cdot,x)](\xi)$ can be extended continuously in ξ on the closed ball $\overline{B}_M$, see page 41 of [57]. Furthermore, by the maximum principle and (3.84), we have

$$|\widetilde{\mathbf{c}}(\xi,x)| \leq 2\delta, \quad \forall \xi, x \in \mathbb{R}^n. \tag{3.86}$$

Now, for $\xi, x \in \mathbb{R}^n$, it follows from (3.83) and (3.85) that

$$\begin{aligned}
\mathbf{a}(\xi,x) &= \mathbf{b}(\xi,x) + (\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}\mathbf{c}(\xi,x) \\
&= \mathbf{b}(\xi,x) + (\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}\widetilde{\mathbf{c}}(\xi,x) \\
&\quad + (\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}\left(\mathbf{c}(\xi,x) - \widetilde{\mathbf{c}}(\xi,x)\right) \\
&= \mathbf{b}(\xi,x) + (\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}\widetilde{\mathbf{c}}(\xi,x) \\
&\quad + (\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}\chi_{\{\xi\in\mathbb{R}^n:|\xi|<M\}}\left(\mathbf{c}(\xi,x) - \widetilde{\mathbf{c}}(\xi,x)\right),
\end{aligned} \tag{3.87}$$

where $\chi_{\{\xi\in\mathbb{R}^n:|\xi|<M\}}$ is the characteristic function of the set $\{\xi \in \mathbb{R}^n : |\xi| < M\}$.

We are now ready to find a new regular function with $p(x)$-growth as follows.

Proposition 3.3.9. *Let $u \in W_0^{1,p(\cdot)}(\Omega)$ be a weak solution to (3.75). Assume that $\mathbf{a}(\xi,x)$ is asymptotically δ-regular with $\mathbf{b}(\xi,x)$ satisfying (3.76), (3.77) and the (δ,R)-vanishing condition (3.78). Define a function $\widetilde{\mathbf{b}}(\xi,x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by*

$$\widetilde{\mathbf{b}}(\xi,x) := \mathbf{b}(\xi,x) + (\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}\widetilde{\mathbf{c}}(Du(x),x), \tag{3.88}$$

where $\widetilde{\mathbf{c}}(\xi,x)$ is given as (3.85). Then we have

1. *$\widetilde{\mathbf{b}}(\xi,x)$ is a regular function with $p(x)$-growth, if $0 < \delta \leq \min\left\{\frac{\lambda}{4(\gamma_2-1)}, \frac{1}{8}\right\}$.*
2. *$\widetilde{\mathbf{b}}(\xi,x)$ is $(5\delta,R)$-vanishing.*

Proof of Proposition 3.3.9. (1) Let $0 < \delta \leq \min\left\{\frac{\lambda}{4(\gamma_2-1)}, \frac{1}{8}\right\}$. We observe from (3.88) and (3.86) that

$$|\widetilde{\mathbf{b}}(\xi,x)| \le |\mathbf{b}(\xi,x)| + \frac{1}{4}(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}. \tag{3.89}$$

Let us note carefully

$$\begin{aligned} D_\xi\widetilde{\mathbf{b}}(\xi,x) &= D_\xi\mathbf{b}(\xi,x) + \widetilde{\mathbf{c}}(Du(x),x) \otimes D_\xi(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}} \\ &= D_\xi\mathbf{b}(\xi,x) + \widetilde{\mathbf{c}}(Du(x),x) \otimes \left((p(x)-1)(\mu^2+|\xi|^2)^{\frac{p(x)-3}{2}}\xi\right), \end{aligned} \tag{3.90}$$

and so

$$\begin{aligned} |D_\xi\widetilde{\mathbf{b}}(\xi,x)| &\le |D_\xi\mathbf{b}(\xi,x)| + \frac{1}{4}(p(x)-1)(\mu^2+|\xi|^2)^{\frac{p(x)-3}{2}}|\xi| \\ &\le |D_\xi\mathbf{b}(\xi,x)| + \frac{1}{4}(\gamma_2-1)(\mu^2+|\xi|^2)^{\frac{p(x)-2}{2}}. \end{aligned} \tag{3.91}$$

Hence, we obtain from (3.89), (3.91) and (3.76) that

$$\begin{aligned} &|\widetilde{\mathbf{b}}(\xi,x)| + (\mu^2+|\xi|^2)^{\frac{1}{2}}|D_\xi\widetilde{\mathbf{b}}(\xi,x)| \\ &\quad\le |\mathbf{b}(\xi,x)| + (\mu^2+|\xi|^2)^{\frac{1}{2}}|D_\xi\mathbf{b}(\xi,x)| + \frac{\gamma_2}{4}(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}} \\ &\quad\le \Lambda(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}} + \frac{\gamma_2}{4}(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}} = \widetilde{\Lambda}(\mu^2+|\xi|^2)^{\frac{p(x)-1}{2}}, \end{aligned} \tag{3.92}$$

where $\widetilde{\Lambda} = \Lambda + \frac{\gamma_2}{4}$.

On the other hand, we conclude from (3.90), (3.77), (3.86) and (3.72) that

$$\begin{aligned} &D_\xi\widetilde{\mathbf{b}}(\xi,x)\eta\cdot\eta \\ &\quad= D_\xi\mathbf{b}(\xi,x)\eta\cdot\eta + (p(x)-1)(\mu^2+|\xi|^2)^{\frac{p(x)-3}{2}}\left(\widetilde{\mathbf{c}}(Du(x),x)\otimes\xi\right)\eta\cdot\eta \\ &\quad\ge \lambda(\mu^2+|\xi|^2)^{\frac{p(x)-2}{2}}|\eta|^2 - 2\delta(p(x)-1)(\mu^2+|\xi|^2)^{\frac{p(x)-2}{3}}|\xi||\eta|^2 \\ &\quad\ge \lambda(\mu^2+|\xi|^2)^{\frac{p(x)-2}{2}}|\eta|^2 - 2\delta(\gamma_2-1)(\mu^2+|\xi|^2)^{\frac{p(x)-2}{2}}|\eta|^2 \\ &\quad= (\lambda-2(\gamma_2-1)\delta)(\mu^2+|\xi|^2)^{\frac{p(x)-2}{2}}|\eta|^2 \\ &\quad\ge \frac{\lambda}{2}(\mu^2+|\xi|^2)^{\frac{p(x)-2}{2}}|\eta|^2, \end{aligned} \tag{3.93}$$

since $0 < \delta \leq \dfrac{\lambda}{4(\gamma_2 - 1)}$. The assertion (1) follows from (3.92) and (3.93).

(2) Let $0 < r \leq R$ and $y \in \mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$, we see from (3.88) that

$$\frac{\widetilde{\mathbf{b}}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} = \frac{\mathbf{b}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} + \widetilde{\mathbf{c}}(Du(x), x). \tag{3.94}$$

Then we deduce from (3.86) that

$$\begin{aligned}
&\left| \frac{\widetilde{\mathbf{b}}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} - \overline{\left(\frac{\widetilde{\mathbf{b}}(\xi, \cdot)}{(\mu^2 + |\xi|^2)^{\frac{p(\cdot)-1}{2}}} \right)}_{B_r(y)} \right| \\
&\quad \leq \left| \frac{\mathbf{b}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} - \overline{\left(\frac{\mathbf{b}(\xi, \cdot)}{(\mu^2 + |\xi|^2)^{\frac{p(\cdot)-1}{2}}} \right)}_{B_r(y)} \right| \\
&\qquad + |\widetilde{\mathbf{c}}(Du(x), x)| + \left| \overline{(\widetilde{\mathbf{c}}(Du(\cdot), \cdot))}_{B_r(y)} \right| \\
&\quad \leq \left| \frac{\mathbf{b}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} - \overline{\left(\frac{\mathbf{b}(\xi, \cdot)}{(\mu^2 + |\xi|^2)^{\frac{p(\cdot)-1}{2}}} \right)}_{B_r(y)} \right| + 4\delta.
\end{aligned} \tag{3.95}$$

Consequently, we have

$$\begin{aligned}
\Theta(\widetilde{\mathbf{b}}, B_r(y))(x) &= \sup_{\xi \in \mathbb{R}^n} \left| \frac{\widetilde{\mathbf{b}}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} - \overline{\left(\frac{\widetilde{\mathbf{b}}(\xi, \cdot)}{(\mu^2 + |\xi|^2)^{\frac{p(\cdot)-1}{2}}} \right)}_{B_r(y)} \right| \\
&\leq \sup_{\xi \in \mathbb{R}^n} \left| \frac{\mathbf{b}(\xi, x)}{(\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}} - \overline{\left(\frac{\mathbf{b}(\xi, \cdot)}{(\mu^2 + |\xi|^2)^{\frac{p(\cdot)-1}{2}}} \right)}_{B_r(y)} \right| + 4\delta \\
&= \Theta(\mathbf{b}, B_r(y))(x) + 4\delta.
\end{aligned} \tag{3.96}$$

Since $\mathbf{b}(\xi, x)$ is (δ, R)-vanishing, we obtain

$$\begin{aligned}
&\sup_{0<r\leq R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} |\Theta(\widetilde{\mathbf{b}}, B_r(y))(x)| \, dx \\
&\quad \leq \sup_{0<r\leq R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} |\Theta(\mathbf{b}, B_r(y))(x)| \, dx + 4\delta
\end{aligned}$$

$$\leq \delta + 4\delta = 5\delta, \tag{3.97}$$

which proves our assertion that $\widetilde{\mathbf{b}}(\xi, x)$ is $(5\delta, R)$-vanishing. $\square$

In view of Proposition 3.3.9, we will consider an auxiliary problem with the regular vector-valued function (3.88) to prove our main result.

Proof of Theorem 3.3.7. Let $\delta_0 > 0$ be the universal constant which is given in Lemma 3.3.6 and set

$$\delta := \frac{1}{5} \min \left\{ \delta_0,\ \frac{\lambda}{4(\gamma_2 - 1)},\ \frac{1}{8} \right\} > 0.$$

Assume that $\mathbf{a}(\xi, x)$ is asymptotically δ-regular with $\mathbf{b}(\xi, x)$ satisfying (3.76), (3.77) and the (δ, R)-vanishing condition (3.78), and let $u \in W_0^{1,p(\cdot)}(\Omega)$ be a weak solution to the problem (3.75).

We first claim that $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution to

$$\operatorname{div} \widetilde{\mathbf{b}}(Du, x) = \operatorname{div} \left(|G|^{p(x)-2} G\right) \quad \text{in} \quad \Omega, \tag{3.98}$$

where $\widetilde{\mathbf{b}}(\xi, x)$ is defined by (3.88) and $G \in L^{p(\cdot)}(\Omega; \mathbb{R}^n)$ is defined by

$$\begin{aligned} |G|^{p(x)-2} G = & |F|^{p(x)-2} F \\ & + (\mu^2 + |Du|^2)^{\frac{p(x)-1}{2}} \chi_{\{|Du|<M\}} \left(\widetilde{\mathbf{c}}(Du, x) - \mathbf{c}(Du, x)\right). \end{aligned} \tag{3.99}$$

Here we write $\chi_{\{|Du|<M\}} = \chi_{\{x \in \Omega : |Du(x)|<M\}}$ to mean the characteristic function of the set $\{x \in \Omega : |Du(x)| < M\}$. Indeed, it follows from (3.87) and (3.88) that

$$\begin{aligned} \operatorname{div} \mathbf{a}(Du, x) = & \operatorname{div} \widetilde{\mathbf{b}}(Du, x) \\ & + \operatorname{div} \left((\mu^2 + |Du|^2)^{\frac{p(x)-1}{2}} \chi_{\{|Du|<M\}} \left(\mathbf{c}(Du, x) - \widetilde{\mathbf{c}}(Du, x)\right) \right) \end{aligned}$$

in the weak sense. Thus (3.75) implies that $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution to

$$\begin{aligned} & \operatorname{div} \widetilde{\mathbf{b}}(Du, x) \\ & \quad = \operatorname{div} \left(|F|^{p(x)-2} F - (\mu^2 + |Du|^2)^{\frac{p(x)-1}{2}} \chi_{\{|Du|<M\}} \left(\mathbf{c}(Du, x) - \widetilde{\mathbf{c}}(Du, x)\right) \right) \end{aligned}$$

$$= \operatorname{div}\,(|G|^{p(x)-2}G) \qquad \text{in} \;\; \Omega. \tag{3.100}$$

We next show that $G \in L^{p(\cdot)}(\Omega;\mathbb{R}^n)$ with the estimate

$$\int_\Omega |G|^{p(x)}dx \le c\int_\Omega \left[|F|^{p(x)}+1\right]dx, \tag{3.101}$$

where $c = c(\gamma_1,\gamma_2,\varphi(\cdot))$ is a positive constant. Observe from (3.86) that

$$|\widetilde{\mathbf{c}}(Du,x)| \le 2\delta < \frac{1}{4}, \tag{3.102}$$

uniformly with respect to $x \in \Omega$. Also, by (3.80),

$$\begin{aligned}\left|(\mu^2+|Du|^2)^{\frac{p(x)-1}{2}}\mathbf{c}(Du,x)\right| &= |\mathbf{a}(Du,x)-\mathbf{b}(Du,x)|\\ &\le \varphi(|Du|)(1+|Du|^{p(x)-1})\\ &\le \|\varphi\|_\infty\,(1+|Du|^{p(x)-1}),\end{aligned}$$

and hence

$$\begin{aligned}\left|(\mu^2+|Du|^2)^{\frac{p(x)-1}{2}}\chi_{\{|Du|<M\}}\mathbf{c}(Du,x)\right| &\le \|\varphi\|_\infty\,(1+|Du|^{p(x)-1})\chi_{\{|Du|<M\}}\\ &\le \|\varphi\|_\infty\,(1+M^{p(x)-1})\\ &\le 2\,\|\varphi\|_\infty\,M^{p(x)-1}.\end{aligned} \tag{3.103}$$

From (3.99), we have

$$|G|^{p(x)-1} = \left||F|^{p(x)-2}F+(\mu^2+|Du|^2)^{\frac{p(x)-1}{2}}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)\right|. \tag{3.104}$$

Therefore, we obtain from (3.102), (3.103), (3.104) and (3.72) that

$$\begin{aligned}|G|^{p(x)} &= \left||F|^{p(x)-2}F+(\mu^2+|Du|^2)^{\frac{p(x)-1}{2}}\chi_{\{|Du|<M\}}\left(\widetilde{\mathbf{c}}(Du,x)-\mathbf{c}(Du,x)\right)\right|^{\frac{p(x)}{p(x)-1}}\\ &\le c\left(\left(|F|^{p(x)-1}\right)^{\frac{p(x)}{p(x)-1}}+\left(\left(\|\varphi\|_\infty+1\right)M^{p(x)-1}\right)^{\frac{p(x)}{p(x)-1}}\right)\\ &= c\left(|F|^{p(x)}+(\|\varphi\|_\infty+1)^{\frac{p(x)}{p(x)-1}}\,M^{p(x)}\right)\\ &\le c\left(|F|^{p(x)}+(\|\varphi\|_\infty+1)^{\frac{\gamma_2}{\gamma_1-1}}\,M^{\gamma_2}\right),\end{aligned} \tag{3.105}$$

where $c = c(\gamma_1, \gamma_2)$ is a positive constant. Since $\Omega \subset \mathbb{R}^n$ is a bounded domain and $F \in L^{p(\cdot)}(\Omega; \mathbb{R}^n)$, we get $G \in L^{p(\cdot)}(\Omega; \mathbb{R}^n)$. Furthermore, (3.105) gives

$$\begin{aligned}\int_\Omega |G|^{p(x)} dx &\le c \int_\Omega \left[|F|^{p(x)} + (\|\varphi\|_\infty + 1)^{\frac{\gamma_2}{\gamma_1 - 1}} M^{\gamma_2} \right] dx \\ &\le c \int_\Omega \left[|F|^{p(x)} + 1 \right] dx,\end{aligned}$$

where $c = c(\gamma_1, \gamma_2, \varphi(\cdot))$ is a positive constant.

In addition, for $q(\cdot)$ satisfying (3.74), we have

$$\begin{aligned}|G|^{p(x)q(x)} &\le c \left(|F|^{p(x)q(x)} + (\|\varphi\|_\infty + 1)^{\frac{p(x)q(x)}{p(x)-1}} M^{p(x)q(x)} \right) \\ &\le c \left(|F|^{p(x)q(x)} + (\|\varphi\|_\infty + 1)^{\frac{\gamma_2\gamma_4}{\gamma_1 - 1}} M^{\gamma_2\gamma_4} \right) \qquad (3.106)\end{aligned}$$

in a similar manner as in (3.105), where $c = c(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is a positive constant. Then it follows from the assumption $|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega)$ and (3.106) that $|G|^{p(\cdot)} \in L^{q(\cdot)}(\Omega)$ with the estimate

$$\begin{aligned}\int_\Omega |G|^{p(x)q(x)} dx &\le c \int_\Omega \left[|F|^{p(x)q(x)} + (\|\varphi\|_\infty + 1)^{\frac{\gamma_2\gamma_4}{\gamma_1 - 1}} M^{\gamma_2\gamma_4} \right] dx \\ &\le c \int_\Omega \left[|F|^{p(x)q(x)} + 1 \right] dx, \qquad (3.107)\end{aligned}$$

where $c = c(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \varphi(\cdot))$ is a positive constant.

Now we consider the auxiliary problem

$$\left\{ \begin{array}{rcll} \operatorname{div} \widetilde{\mathbf{b}}(Dv, x) & = & \operatorname{div} (|G|^{p(x)-2} G) & \text{in} \quad \Omega, \\ v & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \qquad (3.108)$$

We note from Proposition 3.3.9 that $\widetilde{\mathbf{b}}(\xi, x)$ is a regular function with $p(x)$-growth and $(5\delta, R)$-vanishing, and so (δ_0, R)-vanishing. Applying Lemma 3.3.6 to $|G|^{p(\cdot)} \in L^{q(\cdot)}(\Omega)$ and $\widetilde{\mathbf{b}}(\xi, x)$, we discover that $|Dv|^{p(\cdot)} \in L^{q(\cdot)}(\Omega)$ with the estimate

$$\int_\Omega |Dv|^{p(x)q(x)} dx \le c \left(\int_\Omega |G|^{p(x)q(x)} dx + 1 \right)^{\frac{n(\gamma_4 - 1) + \gamma_4}{\gamma_3}}, \qquad (3.109)$$

where $c = c(\mathtt{data}, \omega(\cdot), \rho(\cdot), R, \Omega)$ is a positive constant.

On the other hand, it follows from (3.98) and the uniqueness of the solution to the problem (3.108) that $u = v$ in Ω. Therefore, (3.107) and (3.109) yield

$$\begin{aligned}\int_\Omega |Du|^{p(x)q(x)}dx = \int_\Omega |Dv|^{p(x)q(x)}dx &\le c\left(\int_\Omega |G|^{p(x)q(x)}dx + 1\right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}} \\ &\le c\left(\int_\Omega |F|^{p(x)q(x)}dx + 1\right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}}\end{aligned}$$

for some positive constant $c = c(\mathtt{data}, \varphi(\cdot), \omega(\cdot), \rho(\cdot), R, \Omega)$. This completes the proof. □

3.4 $W^{2,p}$-estimates for solutions to asymptotically elliptic equations in nondivergence form

In this section, we study the global $W^{2,p}$ estimate for viscosity solutions to fully nonlinear, asymptotically elliptic equations of the form

$$F(D^2u, x) = f \quad \text{in } \ \Omega, \tag{3.110}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$.

We aim to obtain the global $W^{2,p}$ estimate for a large class of elliptic equations. To this end, we only assume that $F(M, x)$ is asymptotically elliptic, that is, it has a very general behavior near the infinity.

3.4.1 Hypotheses and main results

Let Ω be a bounded open domain in $\mathbb{R}^n$ with $n \ge 2$. We consider the following fully nonlinear elliptic equations of the form

$$\begin{cases} F(D^2u, x) &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.111}$$

where $x \in \Omega$ and u, f are functions defined in Ω. $F(M, x)$ is a real valued function defined on $S(n) \times \Omega$ where $S(n)$ is the space of real $n \times n$ symmetric matrices. Throughout this section we deal with asymptotically elliptic equations. To introduce the definition that $F(M,x)$ is asymptotically elliptic, we recall the uniformly elliptic condition.

Definition 3.4.1. $G(M,x) : S(n) \times \Omega \to \mathbb{R}$ *is uniformly elliptic if there exist constants* $0 < \lambda \leq \Lambda < \infty$ *(called ellipticity constants) such that for any* $M \in S(n)$ *and* $x \in \Omega$*,*

$$\lambda \|N\| \leq G(M+N,x) - G(M,x) \leq \Lambda \|N\|, \quad \forall N \geq 0. \tag{3.112}$$

Here we write $N \geq 0$ whenever N is a non-negative definite symmetric matrix. $\|N\|$ denotes the (L^2, L^2)-norm of N, that is, $\|N\| = \sup\limits_{|x|=1} |Nx|$. Therefore $\|N\|$ is equal to the maximum eigenvalue of N whenever $N \geq 0$. Note that if $G(M,x)$ is uniformly elliptic, then it is monotone increasing and Lipschitz in $M \in S(n)$.

This section concerns $W^{2,p}$ estimates for solutions to asymptotically linear elliptic equations and asymptotically fully nonlinear elliptic equations. To do this, we first introduce the asymptotically linear operators.

Definition 3.4.2. $F(M,x)$ *is asymptotically linear if there exist a uniformly elliptic operator* $G(M,x) = a_{ij}(x)M_{ij}$ *and a bounded function* $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim\limits_{r\to\infty} \omega(r) = 0$ *such that*

$$|F(M,x) - G(M,x)| \leq \omega\left(\|M\|\right)\left(1 + \|M\|\right) \tag{3.113}$$

for all $M \in S(n)$ *and* $x \in \Omega$*.*

It is easy to check that $G(M,x) = a_{ij}(x)M_{ij}$ is uniformly elliptic if and only if the matrix $(a_{ij}(x))_{i,j=1}^n$ is uniformly elliptic in the sense that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}^n$$

for some constants $0 < \lambda \leq \Lambda < \infty$.

We remark that the condition (3.113) implies

$$\lim_{\|M\|\to\infty} \frac{F(M,x) - a_{ij}(x)M_{ij}}{\|M\|} = 0, \tag{3.114}$$

uniformly with respect to $x \in \Omega$. The asymptotically linear condition means that the operator is almost linear near the infinity as (3.114). In general,

an asymptotically linear operator is neither uniformly elliptic nor linear. For example, if $(a_{ij}(x))_{i,j=1}^n$ is uniformly elliptic, then $F(M,x) = a_{ij}(x)M_{ij} + \sin(M_{11}^2)$ is asymptotically linear. But it is neither uniformly elliptic nor linear. Moreover it is neither convex nor concave in M.
The second example is a combination of the Monge-Ampère operator and the Laplace operator, which is of the form

$$F(D^2u) = \begin{cases} \det D^2u & \text{if} \quad \|D^2u\| < K, \\ \triangle u & \text{if} \quad \|D^2u\| \geq K, \end{cases}$$

for some large constant $K > 1$, that is, $F(M) = (\det M)\chi_{\{M\in S(n):\|M\|<K\}} + (M_{11}+M_{22}+\cdots+M_{nn})(1-\chi_{\{M\in S(n):\|M\|<K\}})$ where $\chi_{\{M\in S(n):\|M\|<K\}}$ denotes the characteristic function of the set $\{M \in S(n) : \|M\| < K\}$.

We next recall the definition of viscosity solutions of fully nonlinear equations. Since we will deal with the fully nonlinear equation (3.111) without the continuity assumption on f, we mainly consider $W^{2,p}$-viscosity solutions.

Definition 3.4.3. *Let $F(M,x)$ be continuous in M and measurable in x. We assume $f \in L^p_{loc}(\Omega)$ for $p > \frac{n}{2}$. We say that a continuous function u is a $W^{2,p}$-viscosity subsolution (resp. supersolution) of (3.111) if for all $\varphi \in W^{2,p}(B_r(x_0))$ whenever $B_r(x_0) \subset \Omega$, $\varepsilon > 0$ and*

$$F(D^2\varphi(x),x) \leq f(x) - \varepsilon$$
$$(\textit{resp. } F(D^2\varphi(x),x) \geq f(x) + \varepsilon)$$

almost everywhere in $B_r(x_0)$, then $u - \varphi$ cannot attain a local maximum (resp. minimum) at x_0. u is called a $W^{2,p}$-viscosity solution if it is both a $W^{2,p}$-viscosity subsolution and a $W^{2,p}$-viscosity supersolution.

The restriction $p > \frac{n}{2}$ guarantees that the test function φ is continuous and pointwisely differentiable a.e. Thus the derivatives of φ appearing in Definition 3.4.3 have a pointwise sense as well as a distributional sense. For more properties about $W^{2,p}$-viscosity solutions, we refer to [34, 41].

Now we introduce the function β_G in order to measure the oscillation of $G(M,x)$ in the variable x.

Definition 3.4.4. *Let $G : S(n) \times \Omega \to \mathbb{R}$ and let $x_0 \in \Omega$ be fixed. For $x \in \Omega$, we define*

$$\beta_G(x,x_0) := \sup_{M\in S(n)\setminus\{0\}} \frac{|G(M,x) - G(M,x_0)|}{\|M\|}. \tag{3.115}$$

We remark that when $G(M,x) = a_{ij}(x)M_{ij}$, that is to say G is linear, the above function β_G measures the oscillation of the coefficient matrix $A(x) = (a_{ij}(x))_{i,j=1}^n$. Hence

$$\sup_{0<r\leq R}\sup_{x_0\in\mathbb{R}^n}\left(\fint_{B_r(x_0)}\beta_G(x,x_0)^n dx\right)^{\frac{1}{n}}$$

is comparable to

$$\sup_{0<r\leq R}\sup_{x_0\in\mathbb{R}^n}\left(\fint_{B_r(x_0)}|A(x)-\overline{A}_{B_r(x_0)}|^n dx\right)^{\frac{1}{n}},$$

where $\overline{A}_{B_r(x_0)}$ is the integral average of $A(x)$ over $B_r(x_0)$ as defined by

$$\overline{A}_{B_r(x_0)} = \fint_{B_r(x_0)} A(x)dx = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} A(x)dx.$$

Note that $W^{2,p}$ estimates for $n < p < \infty$ of the types [33, Theorem 7.1] and [122, Theorem 4.5] have been extended by L. Escauriaza, see [52], to the range $n - \varepsilon_0 < p < \infty$, where $\varepsilon_0 > 0$ depends only on $\frac{\Lambda}{\lambda}$ and n.

We now state the first main result which is a global $W^{2,p}$ estimate for viscosity solutions to asymptotically linear elliptic equations.

Theorem 3.4.5. *Let $n - \varepsilon_0 < p < \infty$ and let $f \in L^p(\Omega)$. Assume that $\partial\Omega \in C^{1,1}$. Then there exists a constant $\beta_0 = \beta_0(n,p,\lambda,\Lambda) > 0$ such that if $F(M,x)$ is asymptotically linear with $G(M,x) = a_{ij}(x)M_{ij}$, and satisfies*

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}|A(x)-\overline{A}_{B_r(x_0)}|^n dx\right)^{\frac{1}{n}} \leq \beta_0, \tag{3.116}$$

for any $x_0 \in \Omega$ and $0 < r < R_0$, then a $W^{2,p}$-viscosity solution u to the problem (3.111) satisfies $u \in W^{2,p}(\Omega)$ with the estimate

$$\left\|D^2u\right\|_{L^p(\Omega)} \leq c\left(\|f\|_{L^p(\Omega)} + \|F(0,\cdot)\|_{L^p(\Omega)} + 1\right), \tag{3.117}$$

where c is a positive constant depending on n, p, λ, Λ, R_0, ω, and Ω.

To obtain $W^{2,p}$ estimates for solutions to asymptotically fully nonlinear elliptic equations, first we introduce the asymptotically elliptic operators.

Definition 3.4.6. *$F(M,x)$ is asymptotically elliptic if there exists a uniformly elliptic operator $G(M,x)$ and a bounded function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{r\to\infty} \omega(r) = 0$ such that*

$$0 \le F(M,x) - G(M,x) \le \omega\left(\|M\|\right)\left(1 + \|M\|\right), \tag{3.118}$$

for all $M \in S(n)$ and $x \in \Omega$.

Note that the condition that $0 \le F(M,x) - G(M,x)$ for all $M \in S(n)$ and $x \in \Omega$ is concerned with the convexity of $\widetilde{G}(M,x)$, see Subsection 3.4.3. Also the condition (3.118) implies that

$$\lim_{\|M\|\to\infty} \frac{F(M,x) - G(M,x)}{\|M\|} = 0, \tag{3.119}$$

uniformly with respect to $x \in \Omega$.

The asymptotically elliptic condition is weaker than the uniformly elliptic condition. For example, consider the Bellman operator which is of the form

$$G(M,x) = \inf_{\alpha\in\mathcal{A}} (a_{ij}^{\alpha}(x) M_{ij} - f^{\alpha}(x)),$$

where $\mathcal{A}$ is any set, f^α is a real function in Ω for each $\alpha \in \mathcal{A}$, and $\left(a_{ij}^{\alpha}(x)\right)_{i,j=1}^{n}$ has eigenvalues in $[\lambda, \Lambda]$ $(0 < \lambda \le \Lambda < \infty)$ for each $x \in \Omega$ and $\alpha \in \mathcal{A}$. It is easy to see that this operator is uniformly elliptic. Now let

$$\begin{aligned} F(M,x) &= G(M,x) + \sin^2(|x| M_{11}^3) \\ &= \inf_{\alpha\in\mathcal{A}} (a_{ij}^{\alpha}(x) M_{ij} - f^{\alpha}(x)) + \sin^2(|x| M_{11}^3). \end{aligned}$$

Then $F(M,x)$ is asymptotically elliptic, but it is not uniformly elliptic.

Now we state the second main result which is an optimal $W^{2,p}$ estimate for viscosity solutions to asymptotically fully nonlinear elliptic equations.

Theorem 3.4.7. *Let $n - \varepsilon_0 < p < \infty$ and let $f \in L^p(\Omega)$. Assume that $\partial\Omega \in C^{1,1}$. Then there exists a constant $\beta_0 = \beta_0(n,p,\lambda,\Lambda) > 0$ such that if $F(M,x)$ is asymptotically elliptic with $G(M,x)$ which is convex in M, and satisfies*

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega} \beta_G(x,x_0)^n dx\right)^{\frac{1}{n}} \le \beta_0, \tag{3.120}$$

for any $x_0 \in \Omega$ and $0 < r < R_0$, then a $W^{2,p}$-viscosity solution u to the problem (3.111) satisfies $u \in W^{2,p}(\Omega)$ with the estimate

$$\left\|D^2 u\right\|_{L^p(\Omega)} \leq c \left(\|f\|_{L^p(\Omega)} + \|F(0,\cdot)\|_{L^p(\Omega)} + 1 \right), \tag{3.121}$$

where c is a positive constant depending on n, p, λ, Λ, R_0, ω, and Ω.

3.4.2 Transformation to uniformly elliptic equations

In the following three subsections, for the sake of convenience, we employ the letter c to denote any universal constants which can be explicitly computed in terms of known quantities such as n, p, λ, Λ, ω, and the geometric assumption on Ω, and so c might vary from line to line.

The purpose of this subsection is to transform a given asymptotically linear equation or a given asymptotically elliptic equation into a suitable uniformly elliptic equation. To do this, of course, we use mainly the asymptotically elliptic (or asymptotically linear) condition on $F(M,x)$, see (3.113), and the small oscillation assumption on $G(M,x)$, see (3.116) and (3.120).

Let $F(M,x)$ be asymptotically elliptic (or asymptotically linear) with $G(M,x)$ which is uniformly elliptic and satisfies

$$\left(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_G(x,x_0)^n dx \right)^{\frac{1}{n}} \leq \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$, where $\beta_0 > 0$ will be determined.
Then by Definition 3.4.2 or Definition 3.4.6, we have

$$\lim_{\|M\|\to\infty} \frac{F(M,x) - G(M,x)}{\|M\|} = 0.$$

Now we define $H(M,x)$ by

$$H(M,x) := \frac{F(M,x) - G(M,x)}{\|M\|}, \quad (M \neq 0). \tag{3.122}$$

Then there exists $K > 1$ such that the relation

$$\|M\| \geq K \implies |H(M,x)| \leq \beta_0 \quad \forall x \in \Omega, \tag{3.123}$$

holds.

We next define $\widetilde{H}(M,x)$ by

$$\widetilde{H}(M,x) := \begin{cases} H(M,x) & \text{if} \quad \|M\| \geq K, \\ \frac{\|M\|}{K} H\left(\frac{K}{\|M\|}M, x\right) & \text{if} \quad 0 < \|M\| < K, \\ 0 & \text{if} \quad M = 0. \end{cases} \tag{3.124}$$

Then $\widetilde{H}(M,x)$ is also continuous in M and measurable in x and that

$$\left|\widetilde{H}(M,x)\right| \leq \beta_0, \quad \forall M \in S(n), \tag{3.125}$$

uniformly with respect to $x \in \Omega$ by (3.123).

Now, for $M \neq 0$,

$$\begin{aligned} F(M,x) &= G(M,x) + \|M\| H(M,x) \\ &= G(M,x) + \|M\| \widetilde{H}(M,x) + \|M\| \left(H(M,x) - \widetilde{H}(M,x)\right) \\ &= G(M,x) + \|M\| \widetilde{H}(M,x) \\ &\quad + \|M\| \chi_{\{M \in S(n) : \|M\| < K\}} \left(H(M,x) - \widetilde{H}(M,x)\right), \end{aligned} \tag{3.126}$$

since $\widetilde{H}(M,x) = H(M,x)$ if $\|M\| \geq K$, where $\chi_{\{M \in S(n):\|M\|<K\}}$ denotes the characteristic function of the set $\{M \in S(n) : \|M\| < K\}$. If, at $M = 0$, we define $\|M\| H(M,x)\,|_{M=0} := F(0,x) - G(0,x)$, then the equation (3.126) holds for all $M \in S(n)$.

Let u be a $W^{2,p}$-viscosity solution of (3.111). Define $\widetilde{G} : S(n) \times \Omega \to \mathbb{R}$ by

$$\widetilde{G}(M,x) := G(M,x) + \|M\| \widetilde{H}(D^2u(x), x). \tag{3.127}$$

Then by (3.126) and (3.127), we have

$$F(D^2u,x) = \widetilde{G}(D^2u,x) + \left\|D^2u\right\| \chi_{\{\|D^2u\|<K\}} \left(H(D^2u,x) - \widetilde{H}(D^2u,x)\right) \text{ in } \Omega,$$

where $\chi_{\{\|D^2u\|<K\}} = \chi_{\{x \in \Omega : \|D^2u(x)\| < K\}}$ denotes the characteristic function of the set $\{x \in \Omega : \|D^2u(x)\| < K\}$. Thus (3.111) implies that u is a $W^{2,p}$-viscosity solution of

$$\widetilde{G}(D^2u,x) = f - \left\|D^2u\right\| \chi_{\{\|D^2u\|<K\}} \left(H(D^2u,x) - \widetilde{H}(D^2u,x)\right)$$

$$=: g \qquad \text{in } \Omega. \tag{3.128}$$

We then have the following lemma.

Lemma 3.4.8. *Let u be a $W^{2,p}$-viscosity solution of the problem (3.111). Let g be given by (3.128). If $f \in L^p(\Omega)$, then $g \in L^p(\Omega)$ with the estimate*

$$\|g\|_{L^p(\Omega)} \le c\left(\|f\|_{L^p(\Omega)} + 1\right), \tag{3.129}$$

where $c = c(p, \omega, |\Omega|, \beta_0)$ is a positive constant.

Proof. Note that

$$\left|\widetilde{H}(D^2u, x)\right| \le \beta_0 \le 1, \tag{3.130}$$

uniformly with respect to $x \in \Omega$ by (3.125). Also, by (3.113),

$$\begin{aligned} \left|\left\|D^2u\right\| H(D^2u, x)\right| &= |F(D^2u, x) - G(D^2u, x)| \\ &\le \omega\left(\left\|D^2u\right\|\right)\left(1 + \left\|D^2u\right\|\right) \le \|\omega\|_\infty \left(1 + \left\|D^2u\right\|\right), \end{aligned}$$

and hence

$$\begin{aligned} \left|\left\|D^2u\right\| \chi_{\{\|D^2u\|<K\}} H(D^2u, x)\right| &\le \|\omega\|_\infty \left(1 + \left\|D^2u\right\|\right) \chi_{\{\|D^2u\|<K\}} \\ &\le \|\omega\|_\infty (1 + K) \le 2\|\omega\|_\infty K. \end{aligned} \tag{3.131}$$

Therefore by (3.128), (3.130) and (3.131), we have

$$\begin{aligned} |g| &\le |f| + \left\|D^2u\right\| \chi_{\{\|D^2u\|<K\}} \left|H(D^2u, x) - \widetilde{H}(D^2u, x)\right| \\ &\le |f| + (2\|\omega\|_\infty + 1) K. \end{aligned} \tag{3.132}$$

Therefore, from (3.132), we have $g \in L^p(\Omega)$ since $\Omega \subset \mathbb{R}^n$ is a bounded domain and $f \in L^p(\Omega)$.

Moreover, we observe from (3.132) that

$$\begin{aligned} \|g\|^p_{L^p(\Omega)} &\le c\left(\|f\|^p_{L^p(\Omega)} + |\Omega| (2\|\omega\|_\infty + 1)^p K^p\right) \\ &\le c\left(\|f\|^p_{L^p(\Omega)} + 1\right), \end{aligned}$$

or,

$$\|g\|_{L^p(\Omega)} \le c\left(\|f\|_{L^p(\Omega)} + 1\right),$$

which is (3.129), where $c = c(p, \omega, |\Omega|)$ is a positive constant. $\square$

Lemma 3.4.9. *Let u be a $W^{2,p}$-viscosity solution of the problem (3.111). Assume that $F(M,x)$ is asymptotically elliptic (or asymptotically linear) with $G(M,x)$ satisfying*

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_G(x,x_0)^n dx\right)^{\frac{1}{n}} \leq \beta_0, \tag{3.133}$$

for any $x_0 \in \Omega$ and $0 < r < R_0$. Then we have

1. *$\widetilde{G}(M,x)$ is uniformly elliptic, if $0 < \beta_0 \leq \frac{\lambda}{2}$.*
2. *$\widetilde{G}(M,x)$ satisfies*

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_{\widetilde{G}}(x,x_0)^n dx\right)^{\frac{1}{n}} \leq 3\beta_0, \tag{3.134}$$

for any $x_0 \in \Omega$ and $0 < r < R_0$.

Proof. (1) Let $0 < \beta_0 \leq \frac{\lambda}{2}$. For any $M, N \in S(n)$ with $N \geq 0$, we have

$$\begin{aligned}\widetilde{G}&(M+N,x) - \widetilde{G}(M,x)\\ &= G(M+N,x) - G(M,x) + (\|M+N\| - \|M\|)\,\widetilde{H}(D^2u,x),\end{aligned}$$

according to (3.127). Observe from (3.125) that

$$\left|\widetilde{H}(D^2u,x)\right| \leq \beta_0, \tag{3.135}$$

uniformly with respect to $x \in \Omega$. We note that the following triangle inequality

$$|\|M+N\| - \|M\|| \leq \|N\|. \tag{3.136}$$

Hence, by (3.112), (3.125), (3.135) and (3.136), we find that

$$\widetilde{G}(M+N,x) - \widetilde{G}(M,x) \geq \lambda\|N\| - \beta_0\|N\| = (\lambda - \beta_0)\|N\| \geq \frac{\lambda}{2}\|N\|,$$

and

$$\widetilde{G}(M+N,x)-\widetilde{G}(M,x)\leq\Lambda\left\|N\right\|+\beta_0\left\|N\right\|=(\Lambda+\beta_0)\left\|N\right\|\leq\left(\Lambda+\frac{\lambda}{2}\right)\left\|N\right\|,$$

since $0<\beta_0\leq\frac{\lambda}{2}$. That is,

$$\widetilde{\lambda}\left\|N\right\|\leq\widetilde{G}(M+N,x)-\widetilde{G}(M,x)\leq\widetilde{\Lambda}\left\|N\right\|, \tag{3.137}$$

where $\widetilde{\lambda}=\frac{\lambda}{2}$ and $\widetilde{\Lambda}=\Lambda+\frac{\lambda}{2}$. Therefore the assertion (1) now follows from (3.137).

(2) Let $x_0\in\Omega$ and $0<r<R_0$. For any $x\in B_r(x_0)\cap\Omega$, (3.127) and (3.135) imply

$$\begin{aligned}|\widetilde{G}(M,x)-\widetilde{G}(M,x_0)|&\leq|G(M,x)-G(M,x_0)|+\beta_0\left\|M\right\|+\beta_0\left\|M\right\|\\&=|G(M,x)-G(M,x_0)|+2\beta_0\left\|M\right\|,\end{aligned} \tag{3.138}$$

and so

$$\begin{aligned}\beta_{\widetilde{G}}(x,x_0)&=\sup_{M\in S(n)\setminus\{0\}}\frac{|\widetilde{G}(M,x)-\widetilde{G}(M,x_0)|}{\|M\|}\\&\leq\sup_{M\in S(n)\setminus\{0\}}\frac{|G(M,x)-G(M,x_0)|}{\|M\|}+2\beta_0\\&=\beta_G(x,x_0)+2\beta_0.\end{aligned} \tag{3.139}$$

Therefore, by (3.139), (3.133) and the Minkowski inequality, we have

$$\begin{aligned}&\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_{\widetilde{G}}(x,x_0)^n dx\right)^{\frac{1}{n}}\\&\leq\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_G(x,x_0)^n dx\right)^{\frac{1}{n}}+2\beta_0\\&\leq\beta_0+2\beta_0=3\beta_0.\end{aligned} \tag{3.140}$$

Then the assertion (2) follows from (3.140). □

In view of Lemma 3.4.9, the asymptotically elliptic (or asymptotically linear) equation (3.111) turns out to be a uniformly elliptic equation (3.128).

First, for the asymptotically linear equation, we will get $C^{1,1}$ estimates for the limiting problem and then prove $W^{2,p}$ estimates (3.117) by using the above transformation. The complete proof of Theorem 3.4.5 will be given in Subsection 3.4.3.
For the asymptotically elliptic equation, Lemma 3.4.8 and the existing theory for fully nonlinear, uniformly elliptic equations, see Lemma 3.4.11, will be employed to finally derive the required estimate (3.121). We return to Subsection 3.4.4 for the complete proof of Theorem 3.4.7.

3.4.3 Proof of Theorem 3.4.5

In this subsection we establish the global $W^{2,p}$ estimates for solutions to asymptotically linear equations. To do this, we first show that the operator $\widetilde{G}(M,x)$ given by (3.127) satisfies the following interior and boundary $C^{1,1}$ estimates for the limiting problem.

Lemma 3.4.10. *Let u be a $W^{2,p}$-viscosity solution of the problem (3.111). Assume that $F(M,x)$ is asymptotically linear with $G(M,x) = a_{ij}(x)M_{ij}$. Let $\widetilde{G}(M,x)$ be given by (3.127) with $0 < \beta_0 \leq \frac{\lambda}{2}$. Then for any $x_0 \in B_2$ and $w_0 \in C(\partial B_2)$, there exists a solution $w \in C^2(B_2) \cap C(\overline{B}_2)$ of*

$$\begin{cases} \widetilde{G}(D^2 w, x_0) &= 0 \quad in \quad B_2, \\ w &= w_0 \quad on \quad \partial B_2, \end{cases} \tag{3.141}$$

such that

$$\|w\|_{C^{1,1}(\overline{B}_1)} \leq c\,\|w_0\|_{L^\infty(\partial B_2)}. \tag{3.142}$$

Additionally, for any $x_0 \in B_2^+ = B_2 \cap \{x_n > 0\}$ and $w_0 \in C(\partial B_2^+)$ with $w_0 = 0$ on $B_2 \cap \{x_n = 0\}$, there exists a solution $w \in C^2(B_2^+) \cap C(\overline{B}_2^+)$ of

$$\begin{cases} \widetilde{G}(D^2 w, x_0) &= 0 \quad in \quad B_2^+, \\ w &= w_0 \quad on \quad \partial B_2^+, \end{cases} \tag{3.143}$$

such that

$$\|w\|_{C^{1,1}(\overline{B}_1^+)} \leq c\,\|w_0\|_{L^\infty(\partial B_2^+)}. \tag{3.144}$$

Proof. First, in view of Lemma 3.4.9 (1), $\widetilde{G}(M,x)$ is uniformly elliptic. From (3.127), we know that

$$\widetilde{G}(M,x) = G(M,x) + \widetilde{H}(D^2 u(x), x)\,\|M\|$$

$$= a_{ij}(x)M_{ij} + \widetilde{H}(D^2u(x), x)\,\|M\|\,. \tag{3.145}$$

Hence $\widetilde{G}(0,\cdot) = 0$. Furthermore, for each $x_0 \in B_2$,

$$\widetilde{G}(M, x_0) = a_{ij}(x_0)M_{ij} + \widetilde{H}(D^2u(x_0), x_0)\,\|M\|$$

is convex if $\widetilde{H}(D^2u(x_0), x_0) \geq 0$, and concave if $\widetilde{H}(D^2u(x_0), x_0) < 0$. Therefore, by [33, Theorem 6.6], we get the interior $C^{1,1}$ estimate (3.142). Additionally, note that the boundary $C^{1,1}$ estimate (3.144) holds when the operator $\widetilde{G}(M, x_0)$ is convex or concave, see [33, Theorem 6.6] and [122, Proposition 4.1]. Hence, by the same token, we have the boundary $C^{1,1}$ estimate (3.144). □

To prove Theorem 3.4.5, we will use the following existing theory for uniformly elliptic equation. Note that we have the estimate (3.146) in the following lemma by using maximum principle for fully nonlinear elliptic equations, see [60, 77, 78].

Lemma 3.4.11. *[122, Theorem 4.5] Let $n - \varepsilon_0 < p < \infty$ and let $f \in L^p(\Omega)$. Assume that $\partial\Omega \in C^{1,1}$. Then there exists a constant $\beta_0 = \beta_0(n, p, \lambda, \Lambda) > 0$ such that if $F(M, x)$ is uniformly elliptic, convex in M, and satisfies*

$$\left(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_F(x, x_0)^n dx \right)^{\frac{1}{n}} \leq \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$, then a $W^{2,p}$-viscosity solution u to the problem (3.111) satisfies $u \in W^{2,p}(\Omega)$ with the estimate

$$\left\|D^2u\right\|_{L^p(\Omega)} \leq c\left(\|f\|_{L^p(\Omega)} + \|F(0,\cdot)\|_{L^p(\Omega)} + 1 \right), \tag{3.146}$$

where $c = c(n, p, \lambda, \Lambda, R_0, \Omega)$ is a positive constant.

Remark 3.4.12. *Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a mollifier with $\phi \geq 0, \int_{\mathbb{R}^n} \phi = 1$ and set $\phi_k(x) := k^n\phi(kx)$ for $k = 1, 2, \cdots$. For $x \in B_1^+$ we consider the following convolution operators*

$$F_k(M, x) := \int_{\mathbb{R}^n} \phi_k(x - y)F(M, y)dy, \quad k = 1, 2, \cdots,$$

where F is extended by 0 outside B_1^+.

In Lemma 3.4.11, the condition that F is convex in M can be replaced by the condition that F_k satisfy the hypothesis on $C^{1,1}$ estimates uniformly in k. See [122, Remark 4.4].

Now we prove Theorem 3.4.5 by using the above lemmas.

Proof of Theorem 3.4.5. Let $\beta_0 > 0$ be the universal constant which is given in Lemma 3.4.11, and let $\beta_1 := \min\left\{\frac{\lambda}{2},\ 1\right\}$. We set

$$\widetilde{\beta}_0 := \frac{1}{3}\min\{\beta_0, \beta_1\} > 0.$$

Now let u be a $W^{2,p}$-viscosity solution of the problem (3.111). Assume that $F(M,x)$ is asymptotically linear with $G(M,x) = a_{ij}(x)M_{ij}$, and satisfies

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_G(x,x_0)^n dx\right)^{\frac{1}{n}} \le \widetilde{\beta}_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$, from the Definition 3.4.4 and the remark just below it.

Then by Lemma 3.4.9, $\widetilde{G}(M,x)$ is uniformly elliptic and satisfies

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_{\widetilde{G}}(x,x_0)^n dx\right)^{\frac{1}{n}} \le 3\widetilde{\beta}_0 \le \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$.

Now let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a mollifier as in Remark 3.4.12. Then for each $k = 1, 2, \cdots$, we have

$$\begin{aligned}
\widetilde{G}_k(M,x) &:= \int_{\mathbb{R}^n}\phi_k(x-y)\widetilde{G}(M,y)dy \\
&= \int_{\mathbb{R}^n}\phi_k(x-y)\left(a_{ij}(y)M_{ij} + \widetilde{H}(D^2u(y),y)\,\|M\|\right)dy \\
&= \left(\int_{\mathbb{R}^n}\phi_k(x-y)a_{ij}(y)dy\right)M_{ij} \\
&\qquad + \left(\int_{\mathbb{R}^n}\phi_k(x-y)\widetilde{H}(D^2u(y),y)dy\right)\|M\| \\
&= a_{ij}^k(x)M_{ij} + \widetilde{H}_k(x)\,\|M\|,
\end{aligned} \tag{3.147}$$

where $a_{ij}^k(x) = \int_{\mathbb{R}^n} \phi_k(x-y)a_{ij}(y)dy$ and $\widetilde{H}_k(x) = \int_{\mathbb{R}^n} \phi_k(x-y)\widetilde{H}(D^2u(y), y)dy$. Here, it is easy to show that the matrices $\left(a_{ij}^k(x)\right)_{i,j=1}^n$ are uniformly elliptic with the same ellipticity constants as $(a_{ij}(x))_{i,j=1}^n$, and that $\left|\widetilde{H}_k(x)\right| \leq \beta_0$ for all $k = 1, 2, \cdots$. Therefore using Lemma 3.4.10, $\widetilde{G}_k$ satisfy the hypothesis on $C^{1,1}$ estimates uniformly in k.

On the other hand, we have $g \in L^p(\Omega)$ from Lemma 3.4.8. We then apply Lemma 3.4.11 to $g \in L^p(\Omega)$ and $\widetilde{G}(M, x)$ to discover $u \in W^{2,p}(\Omega)$ with the estimate

$$\left\|D^2u\right\|_{L^p(\Omega)} \leq c\left(\|g\|_{L^p(\Omega)} + \left\|\widetilde{G}(0, \cdot)\right\|_{L^p(\Omega)} + 1\right), \tag{3.148}$$

where $c = c(n, p, \lambda, \Lambda, R_0, \Omega)$ is a positive constant.

Since $\widetilde{G}(0, x) = G(0, x)$ and $|F(0, x) - G(0, x)| \leq \omega(0)$ from (3.113), we have

$$\left\|\widetilde{G}(0, \cdot)\right\|_{L^p(\Omega)} = \|G(0, \cdot)\|_{L^p(\Omega)} \leq c\left(\|F(0, \cdot)\|_{L^p(\Omega)} + 1\right), \tag{3.149}$$

for some positive constant $c = c(p, \omega, \Omega)$.

But then Lemma 3.4.8, (3.148) and (3.149) imply

$$\left\|D^2u\right\|_{L^p(\Omega)} \leq c\left(\|f\|_{L^p(\Omega)} + \|F(0, \cdot)\|_{L^p(\Omega)} + 1\right), \tag{3.150}$$

for some positive constant $c = c(n, p, \lambda, \Lambda, R_0, \omega, \Omega)$. □

3.4.4 Proof of Theorem 3.4.7

In this subsection we establish the global $W^{2,p}$ estimates for solutions to asymptotically elliptic equations. Using the transformation in Subsection 3.4.2 and Lemma 3.4.11 which is the existing theory for uniformly elliptic equations, we can prove Theorem 3.4.7 as follows.

Proof of Theorem 3.4.7. Let $\beta_0 > 0$ be the universal constant which is given in Lemma 3.4.11, and let $\beta_1 := \min\left\{\frac{\lambda}{2},\ 1\right\}$. We set

$$\widetilde{\beta}_0 := \frac{1}{3}\min\left\{\beta_0, \beta_1\right\} > 0.$$

Now let u be a $W^{2,p}$-viscosity solution of the problem (3.111). Assume that $F(M,x)$ is asymptotically elliptic with $G(M,x)$ which is convex in M and satisfies

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_G(x,x_0)^n dx\right)^{\frac{1}{n}} \le \widetilde{\beta}_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$.

Then by Lemma 3.4.9, $\widetilde{G}(M,x)$ is uniformly elliptic and satisfies

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_{\widetilde{G}}(x,x_0)^n dx\right)^{\frac{1}{n}} \le 3\widetilde{\beta}_0 \le \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$.

Furthermore since $H(M,x) = \dfrac{F(M,x)-G(M,x)}{\|M\|} \ge 0$ for all $M \in S(n)$ and $x \in \Omega$, we have $\widetilde{H}(M,x) \ge 0$ for all $M \in S(n)$ and $x \in \Omega$ from the definition of $\widetilde{H}(M,x)$. Therefore $\widetilde{H}(D^2u(x),x) \ge 0$ for all $x \in \Omega$. Since $G(M,x)$ and $\|M\|$ are convex in M, $\widetilde{G}(M,x) = G(M,x)+\widetilde{H}(D^2u(x),x)\,\|M\|$ is also convex in M.

On the other hand, we have $g \in L^p(\Omega)$ from Lemma 3.4.8. We then apply Lemma 3.4.11 to $g \in L^p(\Omega)$ and $\widetilde{G}(M,x)$ to discover $u \in W^{2,p}(\Omega)$ with the estimate

$$\left\|D^2u\right\|_{L^p(\Omega)} \le c\left(\|g\|_{L^p(\Omega)} + \left\|\widetilde{G}(0,\cdot)\right\|_{L^p(\Omega)} + 1\right), \tag{3.151}$$

where $c = c(n,p,\lambda,\Lambda,R_0,\Omega)$ is a positive constant.

Since $\widetilde{G}(0,x) = G(0,x)$ and $0 \le F(0,x) - G(0,x) \le \omega(0)$ from (3.118), we have

$$\left\|\widetilde{G}(0,\cdot)\right\|_{L^p(\Omega)} = \|G(0,\cdot)\|_{L^p(\Omega)} \le c\left(\|F(0,\cdot)\|_{L^p(\Omega)} + 1\right), \tag{3.152}$$

for some positive constant $c = c(p,\omega,\Omega)$.

Therefore Lemma 3.4.8, (3.151) and (3.152) imply

$$\left\|D^2u\right\|_{L^p(\Omega)} \le c\left(\|f\|_{L^p(\Omega)} + \|F(0,\cdot)\|_{L^p(\Omega)} + 1\right), \tag{3.153}$$

for some positive constant $c = c(n, p, \lambda, \Lambda, R_0, \omega, \Omega)$. $\square$

We have considered the case that the boundary data is zero. One can also obtain $W^{2,p}$ estimates for viscosity solutions to the following Dirichlet problem:

$$\begin{cases} F(D^2u, x) &= f \quad \text{in} \quad \Omega, \\ u &= \varphi \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.154}$$

Now we state the global $W^{2,p}$ estimates for viscosity solutions to asymptotically elliptic equations with non-zero boundary data.

Theorem 3.4.13. *Let $n - \varepsilon_0 < p < \infty$ and let $f \in L^p(\Omega)$, $\varphi \in W^{2,p}(\Omega)$. Assume that $\partial\Omega \in C^{1,1}$. Then there exists a constant $\beta_0 = \beta_0(n, p, \lambda, \Lambda) > 0$ such that if $F(M, x)$ is asymptotically elliptic with $G(M, x)$ which is convex in M, and satisfies*

$$\left(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_G(x, x_0)^n dx \right)^{\frac{1}{n}} \le \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$, then a $W^{2,p}$-viscosity solution u to the problem (3.154) satisfies $u \in W^{2,p}(\Omega)$ with the estimate

$$\left\| D^2u \right\|_{L^p(\Omega)} \le c \left(\|f\|_{L^p(\Omega)} + \left\| D^2\varphi \right\|_{L^p(\Omega)} + \|F(0, \cdot)\|_{L^p(\Omega)} + 1 \right), \tag{3.155}$$

where c is a positive constant depending on n, p, λ, Λ, R_0, ω, and Ω.

Proof. Let $\beta_0 > 0$ be the universal constant which is given in Lemma 3.4.11, and let $\beta_1 := \min\left\{\frac{\lambda}{2},\ 1\right\}$. We set $\widetilde{\beta}_0 := \frac{1}{3}\min\{\beta_0, \beta_1\} > 0$ as the proof of Theorem 3.4.7.

Then by Lemma 3.4.9, $\widetilde{G}(M, x)$ is uniformly elliptic and satisfies

$$\left(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_{\widetilde{G}}(x, x_0)^n dx \right)^{\frac{1}{n}} \le 3\widetilde{\beta}_0 \le \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$. Furthermore $\widetilde{G}(M, x)$ is convex in M from the proof of Theorem 3.4.7.

Now, we recall that u is a $W^{2,p}$-viscosity solution of

$$\begin{cases} \widetilde{G}(D^2u, x) &= g \quad \text{in} \quad \Omega, \\ u &= \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.156}$$

from (3.128). Let $w := u - \varphi$. Then w is a $W^{2,p}$-viscosity solution of

$$\begin{cases} \widehat{G}(D^2w, x) &= \widehat{g} \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.157}$$

where $\widehat{G}(M, x) := \widetilde{G}(M + D^2\varphi, x) - \widetilde{G}(D^2\varphi, x)$ and $\widehat{g}(x) := g(x) - \widetilde{G}(D^2\varphi, x)$. Clearly, $\widehat{G}(M, x)$ is uniformly elliptic, convex in M, and satisfies

$$\left(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_{\widehat{G}}(x, x_0)^n dx \right)^{\frac{1}{n}} \le \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$. Also, since $\widetilde{G}(M, x)$ is uniformly elliptic, we have

$$|\widehat{g}(x)| \le |g(x)| + \left| \widetilde{G}(D^2\varphi, x) \right| \le |g(x)| + \left| \widetilde{G}(0, x) \right| + \widetilde{\Lambda} \left\| D^2\varphi \right\|,$$

where $\widetilde{\Lambda} = \Lambda + \frac{\lambda}{2}$. Therefore $\widehat{g} \in L^p(\Omega)$ by Lemma 3.4.8 and (3.152) with the estimate

$$\begin{aligned} \|\widehat{g}\|_{L^p(\Omega)} &\le c \left(\|g\|_{L^p(\Omega)} + \left\| \widetilde{G}(0, \cdot) \right\|_{L^p(\Omega)} + \left\| D^2\varphi \right\|_{L^p(\Omega)} \right) \\ &\le c \left(\|f\|_{L^p(\Omega)} + \|F(0, \cdot)\|_{L^p(\Omega)} + \left\| D^2\varphi \right\|_{L^p(\Omega)} + 1 \right). \end{aligned} \tag{3.158}$$

Hence we apply Lemma 3.4.11 to $\widehat{g} \in L^p(\Omega)$ and $\widehat{G}(M, x)$ to discover $w \in W^{2,p}(\Omega)$ with the estimate

$$\begin{aligned} \left\| D^2w \right\|_{L^p(\Omega)} &\le c \left(\|\widehat{g}\|_{L^p(\Omega)} + 1 \right) \\ &\le c \left(\|f\|_{L^p(\Omega)} + \|F(0, \cdot)\|_{L^p(\Omega)} + \left\| D^2\varphi \right\|_{L^p(\Omega)} + 1 \right), \end{aligned} \tag{3.159}$$

where $c = c(n, p, \lambda, \Lambda, R_0, \omega, \Omega)$ is a positive constant. Using $u = w + \varphi$, we arrive at the conclusion (3.155). $\square$

Chapter 4

Global gradient estimates for double phase problems

4.1 Global gradient estimates for non-uniformly elliptic equations

In this section, we are concerned with the global gradient estimates for distributional solutions to non-uniformly elliptic problems considered in bounded $C^{1,\beta}$-domains. The equation under consideration is given by

$$\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F\right), \quad (4.1)$$

where $F : \Omega \to \mathbb{R}^n$ is a given vector field and $\Omega \subset \mathbb{R}^n$ is a bounded open domain with $n \geq 2$. As in the rest of the section, we shall assume that the numbers p, q and the coefficient function $a : \Omega \to \mathbb{R}$ satisfy

$$1 < p < q, \quad 0 \leq a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1]. \quad (4.2)$$

In particular, it is of our interest to investigate sharp conditions on p, q, α and a minimal geometric assumption on $\partial\Omega$ under which the natural Calderón-Zygmund type relation

$$|F|^p + a(x)|F|^q \in L^\gamma(\Omega) \implies |Du|^p + a(x)|Du|^q \in L^\gamma(\Omega). \quad (4.3)$$

holds true for every $\gamma \in [1, \infty)$.

4.1.1 Hypotheses and main results

We study a distributional solutions of the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x, Du) &= \operatorname{div} G(x, F) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.4}$$

where $\partial\Omega$ is the boundary of the domain Ω.

The above problem is a generic one whose model is given by (4.1). Throughout this section, $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinearity defined by (1.14) with (4.2). The nonlinearity $A(x, \xi)$ is measurable with respect to x, differentiable with respect to $\xi \neq 0$, and satisfy the following structural conditions with fixed constants $0 < \nu \leq L < +\infty$:

$$|A(x,\xi)| + |\xi||D_\xi A(x,\xi)| \leq L\left(|\xi|^{p-1} + a(x)|\xi|^{q-1}\right), \tag{4.5}$$

$$\nu\left(|\xi|^{p-2} + a(x)|\xi|^{q-2}\right)|\eta|^2 \leq \langle D_\xi A(x,\xi)\eta, \eta\rangle, \tag{4.6}$$

$$|A(x_1,\xi) - A(x_2,\xi)| \leq L|a(x_1) - a(x_2)||\xi|^{q-1} \tag{4.7}$$

for every $x, x_1, x_2 \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$, where the numbers p, q and the coefficient function $a(\cdot)$ are as in (4.2).

On the other hand, a vector field $G : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ in the nonhomogeneous term of (4.4) is assumed to be a Carathéodory function, namely, measurable in x and continuous in ξ, satisfying the following growth condition:

$$|G(x,\xi)| \leq L\left(|\xi|^{p-1} + a(x)|\xi|^{q-1}\right). \tag{4.8}$$

We remark that the structural condition (4.6) implies the following monotonicity property:

$$\widetilde{\nu}\left((|\xi| + |\eta|)^{p-2} + a(x)(|\xi| + |\eta|)^{q-2}\right)|\xi - \eta|^2 \leq \langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle, \tag{4.9}$$

where $\widetilde{\nu}$ is a positive constant depending only on n, p, q and ν. In particular, for the case $2 \leq p < q$, the above monotonicity property can be reduced to

$$\widetilde{\nu}\left(|\xi - \eta|^p + a(x)|\xi - \eta|^q\right) \leq \langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle. \tag{4.10}$$

In the rest of the section we shall use the notation

$$H(x,\xi) := |\xi|^p + a(x)|\xi|^q, \tag{4.11}$$

for $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

We now introduce a distributional solution to (4.4) under consideration.

Definition 4.1.1. *We say that $u \in W_0^{1,1}(\Omega)$ is a distributional solution to (4.4) if it satisfies*

$$\int_\Omega \langle A(x, Du), D\varphi \rangle \, dx = \int_\Omega \langle G(x, F), D\varphi \rangle \, dx, \tag{4.12}$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

We clearly point out that if $u \in W_0^{1,1}(\Omega)$ is a distributional solution to (4.4) with the natural integrability assumption $H(x, Du)$, $H(x, F) \in L^1(\Omega)$, then (4.12) still holds for every function $\varphi \in W_0^{1,1}(\Omega)$ with $H(x, D\varphi) \in L^1(\Omega)$, see Lemma 4.1.3 below. We also present necessary issues of solutions including existence, uniqueness and standard estimate in the next subsection.

We now state the main result in this section. We shall denote

$$\texttt{data} \equiv \texttt{data}(n, p, q, \alpha, \nu, L, \|a\|_{L^\infty(\Omega)}, [a]_{C^{0,\alpha}(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}). \tag{4.13}$$

Theorem 4.1.2. *Let $u \in W_0^{1,1}(\Omega)$ be the distributional solution to (4.4), with*

$$H(x, Du), \ H(x, F) \in L^1(\Omega), \tag{4.14}$$

and under the assumptions (4.2) and (1.13). Suppose that $\partial\Omega \in C^{1,\beta}$ with $\beta \in [\alpha, 1]$, and

$$H(x, F) \in L^\gamma(\Omega) \quad \textit{for some } \gamma \in (1, \infty). \tag{4.15}$$

Then we have

$$H(x, Du) \in L^\gamma(\Omega)$$

with the estimate

$$\left(\int_\Omega [H(x, Du)]^\gamma \, dx \right)^{\frac{1}{\gamma}} \le c \left(\int_\Omega [H(x, F)]^\gamma \, dx \right)^{\frac{1}{\gamma}}, \tag{4.16}$$

where $c = c(\texttt{data}, \gamma, \Omega)$ is a positive constant.

Hereafter, for the sake of convenience, we employ the letter c to denote any universal constants which can be explicitly computed in terms of known quantities such as `data`, γ and the geometric assumption on Ω, and so c might vary from line to line.

4.1.2 Auxiliary lemmas

In this subsection we present some preliminary results regarding testing, solvability and reference problems. We start with a lemma in terms of distributional solutions and test functions, see [40, Proposition 3.1].

Lemma 4.1.3. *Under the assumptions (4.2) and (1.13), let $B \Subset \Omega$ be a ball and let $Y : B \to \mathbb{R}^n$ be a measurable vector filed such that $H(x, Y) \in L^1(B)$ and which is a distributional solution to the equation*

$$\operatorname{div} T(x, Y) = 0 \quad \text{in} \quad B. \tag{4.17}$$

Here we assume that the vector field $T : B \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the growth condition

$$|T(x, \xi)| \leq L\left(|\xi|^{p-1} + a(x)|\xi|^{q-1}\right). \tag{4.18}$$

for every $x \in B$ and $\xi \in \mathbb{R}^n$. Then every function $\varphi \in W_0^{1,1}(B)$ with $H(x, D\varphi) \in L^1(B)$ satisfies

$$\int_B \langle T(x, Y), D\varphi\rangle \, dx = 0. \tag{4.19}$$

The above lemma is crucial for choosing appropriate test functions. Since we deal with global Calderón-Zygmund estimates, we extend the result of Lemma 4.1.3 to the global one. In fact, one can also have the global version of Lemma 4.1.3. To do this, we first introduce the zero extension lemma and the McShane extension lemma.

Lemma 4.1.4. *[5] Let U be a bounded domain in $\mathbb{R}^n$, and let $v \in W_0^{1,t}(U)$ for some $t \geq 1$. Let $\widetilde{v}$ denote the zero extension of v and $\widetilde{Dv}$ denote the zero extension of Dv, that is,*

$$\widetilde{v}(x) := \begin{cases} v(x) & \text{if } x \in U, \\ 0 & \text{if } x \in U^{\mathsf{c}}, \end{cases} \qquad \widetilde{Dv}(x) := \begin{cases} Dv(x) & \text{if } x \in U, \\ 0 & \text{if } x \in U^{\mathsf{c}}. \end{cases}$$

Then $D\widetilde{v} = \widetilde{Dv}$ in the weak sense, and hence $\widetilde{v} \in W^{1,t}(\mathbb{R}^n)$.

Lemma 4.1.5. *[92] Assume $U \subset \mathbb{R}^n$, and let $b : U \to \mathbb{R}$ be a function with $b \in C^{0,\alpha}(U)$. Then there exists an extension $\widetilde{b} : \mathbb{R}^n \to \mathbb{R}$ of b such that $\widetilde{b} \in C^{0,\alpha}(\mathbb{R}^n)$ with $\|\widetilde{b}\|_{L^\infty(\mathbb{R}^n)} = \|b\|_{L^\infty(U)}$ and $[\widetilde{b}]_{C^{0,\alpha}(\mathbb{R}^n)} = [b]_{C^{0,\alpha}(U)}$.*

Remark 4.1.6. *Assume that $H(x, Du) \in L^1(\Omega)$ for $u \in W_0^{1,1}(\Omega)$. Applying Lemma 4.1.4 on u and Lemma 4.1.5 on $a(\cdot) \in C^{0,\alpha}(\Omega)$, we have $\widetilde{H}(x, D\widetilde{u}) \in L^1(\mathbb{R}^n)$, where $\widetilde{H}(x,\xi) := |\xi|^p + \widetilde{a}(x)|\xi|^q$ for $x \in \mathbb{R}^n, \xi \in \mathbb{R}^n$.*

From Lemma 4.1.3 and Remark 4.1.6, we have the following consequence.

Proposition 4.1.7. *Let $u \in W_0^{1,1}(\Omega)$ be a distributional solution to (4.4) with (4.14), under the assumptions (4.2) and (1.13). Then we have*

$$\int_\Omega \langle A(x, Du), D\varphi\rangle \, dx = \int_\Omega \langle G(x, F), D\varphi\rangle \, dx,$$

for every function $\varphi \in W_0^{1,1}(\Omega)$ with $H(x, D\varphi) \in L^1(\Omega)$.

Now we prove an existence result for the Dirichlet problem (4.4).

Theorem 4.1.8. *Assume that (4.2) and (1.13), and suppose that $H(x, F) \in L^1(\Omega)$. Then there exists a unique distributional solution $u \in W_0^{1,1}(\Omega)$ to (4.4) such that $H(x, Du) \in L^1(\Omega)$. Furthermore, the energy estimate*

$$\int_\Omega H(x, Du) \, dx \le c \int_\Omega H(x, F) \, dx, \tag{4.20}$$

holds for a constant $c = c(n, p, q, \nu, L)$.

Proof. We first introduce the Musielak-Orlicz-Sobolev space $W^{1,H}(\Omega)$ to prove the existence of a solution. By abuse of notation, we shall continue to write $H(x,\xi)$ also when $x \in \Omega$ and $\xi \in \mathbb{R}$. We note that the function $H : \Omega \times [0,\infty) \to [0,\infty)$ under consideration is a Musielak-Orlicz function, see [9, 99, 116] for the definition of Musielak-Orlicz functions. The Musielak-Orlicz class $K^H(\Omega)$ is the set of all measurable functions $v : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega H(x, |v(x)|) \, dx < \infty.$$

The Musielak-Orlicz space $L^H(\Omega)$ is the vector space generated by $K^H(\Omega)$, that is, $L^H(\Omega)$ is the smallest linear space containing $K^H(\Omega)$. In the space

$L^H(\Omega)$ the Luxemburg norm $\|\cdot\|_{L^H(\Omega)}$ is defined as

$$\|v\|_{L^H(\Omega)} = \inf\left\{\sigma > 0 : \int_\Omega H\left(x, \frac{|v(x)|}{\sigma}\right) dx \leq 1\right\}.$$

Since $H(x, 2t) \leq 2^q H(x,t)$ for all $x \in \Omega$ and $t \geq 0$, the Musielak-Orlicz function H satisfies the Δ_2-condition. Hence $K^H(\Omega) = L^H(\Omega)$ and $(L^H(\Omega), \|\cdot\|_{L^H(\Omega)})$ is a Banach space.

The Musielak-Orlicz-Sobolev space $W^{1,H}(\Omega)$ is the function space of all measurable functions $v \in L^H(\Omega)$ such that its distributional gradient vector Dv belongs to $L^H(\Omega;\mathbb{R}^n)$. If $v \in W^{1,H}(\Omega)$, we define its norm to be

$$\|v\|_{W^{1,H}(\Omega)} = \|v\|_{L^H(\Omega)} + \|Dv\|_{L^H(\Omega;\mathbb{R}^n)}.$$

The space $W_0^{1,H}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,H}(\Omega)$. For further discussion of the Musielak-Orlicz space and the associated Sobolev space, we refer the reader to [9, 44, 58, 59, 99, 116] and references therein.

Now the absence of Lavrentiev phenomenon discussed in [55] and the result of [116] allow to find a distributional solution $u \in W_0^{1,H}(\Omega)$ to the problem (4.4). It follows immediately that $u \in W_0^{1,1}(\Omega)$ with $H(x, Du) \in L^1(\Omega)$. Furthermore, from Proposition 4.1.7, we can choose u as a test function, that is, we have

$$\int_\Omega \langle A(x, Du), Du\rangle \, dx = \int_\Omega \langle G(x, F), Du\rangle \, dx.$$

By using the monotonicity property (4.9) with $\eta = 0$, the growth condition (4.8) of G and Young's inequality with $\tau \in (0,1)$, we obtain

$$\begin{aligned}\widetilde{\nu}\int_\Omega H(x, Du)\, dx &\leq \int_\Omega \langle A(x, Du), Du\rangle \, dx = \int_\Omega \langle G(x, F), Du\rangle \, dx \\ &\leq L\int_\Omega (|F|^{p-1} + a(x)|F|^{q-1})|Du|\, dx \\ &\leq L\left[\tau\int_\Omega H(x, Du)\, dx + c\tau^{-\frac{1}{p-1}}\int_\Omega H(x, F)\, dx\right].\end{aligned}$$

The energy estimate (4.20) now follows by taking $\tau = \dfrac{\widetilde{\nu}}{2L}$.

We next show the uniqueness of solutions. Suppose that u_1, u_2 are distributional solutions to (4.4) with $H(x, Du_1), H(x, Du_2) \in L^1(\Omega)$. Then we can take $\varphi = u_1 - u_2 \in W_0^{1,1}(\Omega)$ as a test function by Proposition 4.1.7, and hence we obtain

$$\int_\Omega \langle A(x, Du_1) - A(x, Du_2), Du_1 - Du_2 \rangle \, dx = 0 \tag{4.21}$$

Combining (4.21) with the monotonicity property (4.9) yields $Du_1 = Du_2$ in Ω. Since $u_1 = 0 = u_2$ on $\partial\Omega$, we conclude that $u_1 = u_2$ in Ω. □

Next, we discuss the local regularity for homogeneous equations. Colombo and Mingione proved this result in the interior case, see [40, Theorem 3.1]. We present a boundary version of the local regularity result.

Lemma 4.1.9. *Under the assumptions (4.2) and (1.13), consider a function $u \in W^{1,1}(B_{5r}^+)$ with $0 < r \leq 1$, $H(x, Du) \in L^1(B_{5r}^+)$ and*

$$u = 0 \quad on \;\; T_{5r}.$$

Then there exists a unique distributional solution w to

$$\begin{cases} \operatorname{div} A(x, Dw) & = & 0 & in & B_{4r}^+, \\ \hfill w & = & u & on & \partial B_{4r}^+, \end{cases} \tag{4.22}$$

such that $H(x, Dw) \in L^1(B_{4r}^+)$. Moreover, the energy estimate

$$\int_{B_{4r}^+} H(x, Dw) \, dx \leq c \int_{B_{4r}^+} H(x, Du) \, dx, \tag{4.23}$$

holds for a positive constant $c = c(n, p, q, \nu, L)$, and the local regularity result

$$Dw \in L^{\frac{np}{n-2t}}(B_\rho^+) \cap W^{\min\{\frac{2}{p},1\}t,p}(B_\rho^+) \tag{4.24}$$

holds for every $\rho \in (0, 4r)$ and for every $t < \alpha$. In particular, it follows that for every $\rho \in (0, 4r)$,

$$Dw \in L^q(B_\rho^+). \tag{4.25}$$

Proof. We first introduce the following rescaled functions:

$$\bar{u}(x) := \frac{u(rx)}{r}, \quad \bar{w}(x) := \frac{w(rx)}{r}, \quad \bar{a}(x) := a(rx)$$

and

$$\bar{A}(x,\xi) := A(rx,\xi), \quad \bar{H}(x,\xi) := |\xi|^p + \bar{a}(x)|\xi|^q = H(rx,\xi)$$

for $x \in B_5$ and $\xi \in \mathbb{R}^n$. Then it is easy to see that (4.22) is equivalent to

$$\begin{cases} \operatorname{div} \bar{A}(x, D\bar{w}) &= 0 & \text{in} & B_4^+, \\ \bar{w} &= \bar{u} & \text{on} & \partial B_4^+, \end{cases}$$

and that the energy estimate (4.23) is equivalent to

$$\int_{B_4^+} \bar{H}(x, D\bar{w})\,dx \leq c \int_{B_4^+} \bar{H}(x, D\bar{u})\,dx$$

for a positive constant $c = c(n,p,q,\nu,L)$. Therefore, it is sufficient to prove the lemma for $r = 1$. In view of Lemma 4.1.5, we can assume that $a(\cdot) \in C^{0,\alpha}(\overline{B_5^+})$. Denoting $x \equiv (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we consider the reflection operator $i : \mathbb{R}^n \to \mathbb{R}^n$ defined by $i(x', x_n) = (x', -x_n)$. Then we define the function $\widehat{a}(\cdot) : \overline{B_5} \to \mathbb{R}$ by

$$\widehat{a}(x) := \begin{cases} a(x), & x \in \overline{B_5^+}, \\ a(i(x)), & x \in \overline{B_5} \setminus \overline{B_5^+}. \end{cases} \tag{4.26}$$

It is easy to check that $\widehat{a}(\cdot) \geq 0$ and $\widehat{a}(\cdot) \in C^{0,\alpha}(B_5)$ with $\|\widehat{a}\|_{L^\infty(B_5)} = \|a\|_{L^\infty(B_5^+)}$ and $[\widehat{a}]_{C^{0,\alpha}(B_5)} = [a]_{C^{0,\alpha}(B_{5r}^+)}$. We now define the map

$$\widehat{u}(x) := \begin{cases} u(x), & x \in B_5^+, \\ -u(i(x)), & x \in B_5 \setminus B_5^+, \end{cases} \tag{4.27}$$

being the odd extension of u to the whole ball B_5. It follows that $\widehat{u} \in W^{1,1}(B_5)$ with $\widehat{H}(x, D\widehat{u}) \in L^1(B_5)$, where $\widehat{H}(x,\xi) := |\xi|^p + \widehat{a}(x)|\xi|^q$ for $x \in B_5$ and $\xi \in \mathbb{R}^n$. We note that $B_4^+ \Subset B_5$ is a Lipschitz domain and that $\widehat{u} * \phi = 0$ on T_4 for any radially symmetric convolution kernel ϕ. From the absence of Lavrentiev phenomenon discussed in [55, Section 5], we can find a sequence $\{u_j\}_{j=1}^\infty \subset C^\infty(B_4^+)$ such that $u_j \longrightarrow u$ strongly in $W^{1,p}(B_4^+)$, $u_j = 0$ on T_4, and

$$\int_{B_4^+} H(x, Du_j)\,dx \longrightarrow \int_{B_4^+} H(x, Du)\,dx. \tag{4.28}$$

We set

$$\sigma_j := \frac{1}{j + \|Du_j\|^{2q}_{L^{2q-p}(B_4^+)}}, \quad (j = 1, 2, \cdots). \tag{4.29}$$

Then we have

$$\lim_{j\to\infty} \sigma_j \left(1 + \|Du_j\|^{2q-p}_{L^{2q-p}(B_4^+)}\right) = 0. \tag{4.30}$$

We define the new vector fields

$$A_j(x,\xi) := A(x,\xi) + \sigma_j |\xi|^{2q-p-2}\xi, \quad (j = 1, 2, \cdots).$$

Then for each $j \in \mathbb{N}$ we can find the unique weak solution to the problem

$$\begin{cases} \operatorname{div} A_j(x, Dw_j) = 0 & \text{in} \quad B_4^+, \\ \qquad\qquad w_j = u_j & \text{on} \quad \partial B_4^+, \end{cases} \tag{4.31}$$

as the vector field $A_j(x,\xi)$ is coercive and monotone in $W^{1,2q-p}$. By taking $\varphi = w_j - u_j \in W_0^{1,2q-p}(B_4^+)$ as a test function of the weak formulation of (4.31), we get

$$\int_{B_4^+} \langle A_j(x, Dw_j), D(w_j - u_j)\rangle \, dx = 0.$$

By the monotonicity property of $A_j(x,\xi)$, we have

$$\begin{aligned}\int_{B_4^+} &\left[|Dw_j|^p + a(x)|Dw_j|^q + \sigma_j |Dw_j|^{2q-p}\right] dx \\ &\leq c \int_{B_4^+} \left[|Dw_j|^{p-1} + a(x)|Dw_j|^{q-1} + \sigma_j |Dw_j|^{2q-p-1}\right] |Du_j| \, dx,\end{aligned}$$

for some constant $c = c(n, p, q, \nu, L) > 0$. Using Young's inequality, we obtain

$$\begin{aligned}\int_{B_4^+} H(x, Dw_j)\, dx &\leq \int_{B_4^+} \left[H(x, Dw_j) + \sigma_j |Dw_j|^{2q-p}\right] dx \\ &\leq c \int_{B_4^+} \left[H(x, Du_j) + \sigma_j |Du_j|^{2q-p}\right] dx. \end{aligned} \tag{4.32}$$

Therefore, there exist a subsequence, which we still denote by $\{w_j\}_{j=1}^{\infty}$, and a function $w \in u + W_0^{1,p}(B_4^+)$ such that $w_j \rightharpoonup w$ in $W^{1,p}(B_4^+)$. Then it follows from (4.28), (4.30) and the lower semicontinuity in (4.32) that the energy estimate (4.23) holds.

For each $j = 1, 2, \cdots$, we now construct a family of regularized vector fields

$$A_j^{\ell} : \overline{B_3^+} \times \mathbb{R}^n \to \mathbb{R}^n, \quad |\ell| \in (0,1).$$

To do this, we first extend the vector field $A : B_4^+ \times \mathbb{R}^n \to \mathbb{R}^n$ to a new vector field $\widehat{A} : B_4 \times \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\widehat{A}(x,\xi) := \begin{cases} A(x,\xi), & x \in B_4^+, \\ A(i(x),\xi), & x \in B_4 \setminus B_4^+, \end{cases} \tag{4.33}$$

where $i : \mathbb{R}^n \to \mathbb{R}^n$ is the reflection operator. It is easy to check that the vector field $\widehat{A}$ still satisfies structural conditions (4.5)-(4.7), replacing $a(\cdot)$ by $\widehat{a}(\cdot)$ defined in (4.26). Let ϕ be a smooth, positive, radially symmetric convolution kernel supported in B_1 satisfying $\int_{B_1} \phi \, dx = 1$. For $|\ell| \in (0,1)$, we define

$$\begin{aligned} A^{\ell}(x,\xi) &:= \int_{B_1} \widehat{A}(x + |\ell| y, \xi)\, \phi(y)\, dy \\ &= \frac{1}{|\ell|^n} \int_{B_{|\ell|}(x)} \widehat{A}(y,\xi)\, \phi\left(\frac{y-x}{|\ell|}\right) dy \end{aligned}$$

for $x \in B_{4-|\ell|}$ and $\xi \in \mathbb{R}^n$. Then the regularized vector field $A^{\ell} : B_{4-|\ell|} \times \mathbb{R}^n \to \mathbb{R}^n$ with $|\ell| \in (0,1)$ satisfies the following conditions:

$$|A^{\ell}(x,\xi)| + |\xi||D_\xi A^{\ell}(x,\xi)| \le L\left(|\xi|^{p-1} + a^{\ell}(x)|\xi|^{q-1}\right), \tag{4.34}$$

$$\nu\left(|\xi|^{p-2} + a^{\ell}(x)|\xi|^{q-2}\right)|\eta|^2 \le \langle D_\xi A^{\ell}(x,\xi)\eta, \eta\rangle, \tag{4.35}$$

$$|D_x A^{\ell}(x,\xi)| \le cL[\widehat{a}]_{C^{0,\alpha}(B_4)} |\ell|^{\alpha-1} |\xi|^{q-1}, \tag{4.36}$$

$$|A^{\ell}(x_1,\xi) - A^{\ell}(x_2,\xi)| \le cL[\widehat{a}]_{C^{0,\alpha}(B_4)} |\ell|^{\alpha-1} |x_1 - x_2||\xi|^{q-1}, \tag{4.37}$$

$$|A^{\ell}(x,\xi) - \widehat{A}(x,\xi)| \le L[\widehat{a}]_{C^{0,\alpha}(B_4)} |\ell|^{\alpha} |\xi|^{q-1} \tag{4.38}$$

for every $x, x_1, x_2 \in B_{4-|\ell|}$ and $\xi, \eta \in \mathbb{R}^n$, where the positive constant $c = c(n, \|D\phi\|_{L^\infty})$ is independent of ℓ, and $a^{\ell} : B_{r-|\ell|} \to \mathbb{R}$ is defined by

$$a^{\ell}(x) := \int_{B_1} \widehat{a}(x + |\ell| y)\, \phi(y)\, dy = \frac{1}{|\ell|^n} \int_{B_{|\ell|}(x)} \widehat{a}(y)\, \phi\left(\frac{y-x}{|\ell|}\right) dy.$$

Indeed, (4.34) and (4.35) can be directly obtained from the definitions of

$A^\ell(x,\xi)$ and $a^\ell(x)$ with elementary computations. To prove (4.36)-(4.38), we observe from the structural condition of $\widehat{A}$ that

$$\begin{aligned}\left|\widehat{A}(x+|\ell|y,\xi)-\widehat{A}(x,\xi)\right| &\le |\widehat{a}(x+|\ell|y)-\widehat{a}(x)||\xi|^{q-1}\\ &\le L[\widehat{a}]_{C^{0,\alpha}(B_4)}(|\ell||y|)^\alpha|\xi|^{q-1}\\ &\le L[\widehat{a}]_{C^{0,\alpha}(B_4)}|\ell|^\alpha|\xi|^{q-1}\end{aligned}\tag{4.39}$$

for every $|\ell|\in(0,1)$, $x\in B_{4-|\ell|}$, $y\in B_1$ and $\xi\in\mathbb{R}^n$. Then we obtain from (4.39) and the definition of $A^\ell(x,\xi)$ that

$$\begin{aligned}|D_xA^\ell(x,\xi)| &= \left|-\frac{1}{|\ell|^{n+1}}\int_{B_{|\ell|}(x)}\widehat{A}(y,\xi)\,(D_x\phi)\left(\frac{y-x}{|\ell|}\right)dy\right|\\ &= \frac{1}{|\ell|}\left|\int_{B_1}\widehat{A}(x+|\ell|y,\xi)\,(D_y\phi)(y)\,dy\right|\\ &= \frac{1}{|\ell|}\left|\int_{B_1}\left[\widehat{A}(x+|\ell|y,\xi)-\widehat{A}(x,\xi)\right](D_y\phi)(y)\,dy\right|\\ &\le L[\widehat{a}]_{C^{0,\alpha}(B_4)}|\ell|^{\alpha-1}|\xi|^{q-1}\,\|D\phi\|_{L^\infty}\,|B_1|\\ &\le cL[\widehat{a}]_{C^{0,\alpha}(B_4)}|\ell|^{\alpha-1}|\xi|^{q-1},\end{aligned}$$

which is (4.36). The inequality (4.37) follows directly from (4.36). Also we can derive (4.38) as follows:

$$\begin{aligned}|A^\ell(x,\xi)-\widehat{A}(x,\xi)| &= \left|\int_{B_1}\left[\widehat{A}(x+|\ell|y,\xi)-\widehat{A}(x,\xi)\right]\phi(y)\,dy\right|\\ &\le L[\widehat{a}]_{C^{0,\alpha}(B_4)}|\ell|^\alpha|\xi|^{q-1}\int_{B_1}\phi(y)\,dy\\ &= L[\widehat{a}]_{C^{0,\alpha}(B_4)}|\ell|^\alpha|\xi|^{q-1}.\end{aligned}$$

We now define

$$A_j^\ell(x,\xi) := A^\ell(x,\xi)+\sigma_j|\xi|^{2q-p-2}\xi, \quad (j=1,2,\cdots),$$

for $x\in B_{4-|\ell|}$ and $\xi\in\mathbb{R}^n$, where $\sigma_j>0$ is the constant defined in (4.29). We note that, for each $j=1,2,\cdots$ and $|\ell|\in(0,1)$, the vector field A_j^ℓ is coercive and monotone in $W^{1,2q-p}$ and w_j belongs to $W^{1,2q-p}(B_4^+)$. Hence we can find the unique weak solution to the problem

$$\begin{cases} \operatorname{div} A_j^\ell(x, Dv_j^\ell) &= 0 \quad \text{in} \quad B_3^+, \\ v_j^\ell &= w_j \quad \text{on} \quad \partial B_3^+. \end{cases} \tag{4.40}$$

We introduce the auxiliary vector field $V_p : \mathbb{R}^n \to \mathbb{R}^n$ by

$$V_p(\xi) := |\xi|^{\frac{p-2}{2}}\xi, \quad (\xi \in \mathbb{R}^n),$$

and claim that $V_p(Dv_j^\ell) \in W^{1,2}(B_2^+)$ with the estimate

$$\begin{aligned} \int_{B_2^+} |D(V_p(Dv_j^\ell))|^2\,dx &\le c\int_{B_3^+} |Dv_j^\ell|^p\,dx \\ &\quad + c\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2(\alpha-1)} + \sigma_j\right)\int_{B_3^+} |Dv_j^\ell|^{2q-p}\,dx. \end{aligned} \tag{4.41}$$

for some constant $c = c(n, p, q, \nu, L) > 0$. To see this, let $\zeta \in C_0^\infty(B_{\frac{8}{3}})$ be a cut-off function such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in $B_{\frac{7}{3}}$ and $|D\zeta|^2 + |D^2\zeta| \le 10^4$. We consider the finite difference operator $\tau_{s,h}$ for $s \in \{1, \cdots, n-1\}$ and $h \in \mathbb{R}$ with $|h| \in (0, 10^{-4}]$. Since $v_j^\ell = w_j = u_j = 0$ on T_3, we can take $\varphi = \tau_{s,-h}(\zeta^2\tau_{s,h}v_j^\ell) \in W_0^{1,2q-p}(B_3^+)$ as a test function in (4.40). We note from the definition of $a^\ell(\cdot)$ that $\|a^\ell\|_{L^\infty(B_3^+)} \le \|a\|_{L^\infty(B_4^+)}$. By carefully inspecting the computations in the proof of [38, Theorem 5.1], we can obtain the following inequality:

$$\begin{aligned} \int_{B_3^+} \zeta^2|\tau_{s,h}(V_p(Dv_j^\ell))|^2\,dx &\le c|h|^2\int_{B_3^+} |Dv_j^\ell|^p\,dx \\ &\quad + c|h|^2\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2(\alpha-1)} + \sigma_j\right)\int_{B_3^+} |Dv_j^\ell|^{2q-p}\,dx. \end{aligned} \tag{4.42}$$

This allows us to prove the existence of $D_s(V_p(Dv_j^\ell))$, $s = 1, \cdots, n-1$, in $L^2(B_2^+)$. In addition, we have

$$\int_{B_2^+} |Dv_j^\ell|^{p-2}|DD'v_j^\ell|^2\,dx \le c\int_{B_3^+} |Dv_j^\ell|^p\,dx$$

$$+ c\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2(\alpha-1)} + \sigma_j\right)\int_{B_3^+}|Dv_j^\ell|^{2q-p}\,dx, \tag{4.43}$$

where $D' \equiv (D_1, \cdots, D_{n-1})$ denotes the tangential gradient with respect to the hyperplane $\{x_n = 0\}$. On the other hand, we see from the estimate (4.42) with a suitable cut-off function $\zeta \in C_0^\infty(B^+_{\frac{8}{3}})$ and $s = 1, \cdots, n$ that $V_p(Dv_j^\ell)$ is differentiable in the interior, that is, $V_p(Dv_j^\ell) \in W^{1,2}_{\mathrm{loc}}(B_2^+)$. Therefore, the equation (4.40) yields

$$\begin{aligned} D_{\xi_n}((A_j^\ell)_n)(x, Dv_j^\ell)D_{nn}v_j^\ell = &- \sum_{\substack{i,k=1\\(i,k)\neq(n,n)}}^n D_{\xi_k}((A_j^\ell)_i)(x, Dv_j^\ell)D_{ik}v_j^\ell \\ &- \sum_{i=1}^n D_x((A_j^\ell)_i)(x, Dv_j^\ell) \end{aligned}$$

almost everywhere in B_2^+. Multiplying by $D_{nn}v_j^\ell$ and using the structural conditions of $A_j^\ell(x,\xi)$, we get

$$\begin{aligned} &\nu\left(|Dv_j^\ell|^{p-2} + a^\ell(x)|Dv_j^\ell|^{q-2} + \sigma_j|Dv_j^\ell|^{2q-p-2}\right)|D_{nn}v_j^\ell|^2 \\ &\quad \le L\left(|Dv_j^\ell|^{p-2} + a^\ell(x)|Dv_j^\ell|^{q-2} + \sigma_j|Dv_j^\ell|^{2q-p-2}\right)|DD'v_j^\ell||D_{nn}v_j^\ell| \\ &\qquad + cL[a]_{C^{0,\alpha}(B_4^+)}|\ell|^{\alpha-1}|Dv_j^\ell|^{q-1}|D_{nn}v_j^\ell|. \end{aligned}$$

Noting that $|Dv_j^\ell|^{q-1}|D_{nn}v_j^\ell| = |Dv_j^\ell|^{\frac{2q-p}{2}}|Dv_j^\ell|^{\frac{p-2}{2}}|D_{nn}v_j^\ell|$, we obtain from Young's inequality that

$$\begin{aligned} &\left(|Dv_j^\ell|^{p-2} + a^\ell(x)|Dv_j^\ell|^{q-2} + \sigma_j|Dv_j^\ell|^{2q-p-2}\right)|D_{nn}v_j^\ell|^2 \\ &\quad \le c\left(|Dv_j^\ell|^{p-2} + a^\ell(x)|Dv_j^\ell|^{q-2} + \sigma_j|Dv_j^\ell|^{2q-p-2}\right)|DD'v_j^\ell|^2 \\ &\qquad + c[a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2(\alpha-1)}|Dv_j^\ell|^{2q-p}, \end{aligned}$$

and hence

$$|Dv_j^\ell|^{p-2}|D_{nn}v_j^\ell|^2 \le c|Dv_j^\ell|^{p-2}|DD'v_j^\ell|^2 + c[a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2(\alpha-1)}|Dv_j^\ell|^{2q-p}. \tag{4.44}$$

Combining (4.43) and (4.44) gives

$$\int_{B_2^+} |D(V_p(Dv_j^\ell))|^2\,dx \le c\int_{B_2^+} |Dv_j^\ell|^{p-2}|D^2v_j^\ell|^2\,dx$$
$$\le c\int_{B_3^+}|Dv_j^\ell|^p\,dx + c\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2(\alpha-1)} + \sigma_j\right)\int_{B_3^+}|Dv_j^\ell|^{2q-p}\,dx,$$

which is our claim (4.41). We now take $\varphi = v_j^\ell - w_j \in W_0^{1,2q-p}(B_3^+)$ as a test function in (4.31) and (4.40) to discover that

$$\int_{B_3^+}\langle A_j^\ell(x,Dv_j^\ell) - A_j^\ell(x,Dw_j), D(v_j^\ell - w_j)\rangle\,dx$$
$$= \int_{B_3^+}\langle A_j(x,Dw_j) - A_j^\ell(x,Dw_j), D(v_j^\ell - w_j)\rangle\,dx. \quad (4.45)$$

We notice that there exists a constant $c = c(n,p) > 1$ such that

$$c^{-1}(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2 \le |V_p(\xi)-V_p(\eta)|^2 \le c(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2 \quad (4.46)$$

for any $\xi,\eta\in\mathbb{R}^n$. It follows from the structural conditions of A_j and A_j^ℓ, (4.46) and Young's inequality with $\theta\in(0,1)$ that

$$\begin{aligned}
&\widetilde{\nu}\int_{B_3^+}|V_p(Dv_j^\ell)-V_p(Dw_j)|^2\,dx\\
&\quad\le cL[a]_{C^{0,\alpha}(B_4^+)}|\ell|^\alpha\int_{B_3^+}|Dw_j|^{q-1}|Dv_j^\ell - Dw_j|\,dx\\
&\quad\le cL[a]_{C^{0,\alpha}(B_4^+)}|\ell|^\alpha\int_{B_3^+}(|Dv_j^\ell|+|Dw_j|)^{q-1}|Dv_j^\ell - Dw_j|\,dx\\
&\quad\le \theta\int_{B_3^+}(|Dv_j^\ell|+|Dw_j|)^{p-2}|Dv_j^\ell-Dw_j|^2\,dx\\
&\qquad + c\theta^{-1}L^2[a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2\alpha}\int_{B_3^+}(|Dv_j^\ell|+|Dw_j|)^{2q-p}\,dx\\
&\quad\le c\theta\int_{B_3^+}|V_p(Dv_j^\ell)-V_p(Dw_j)|^2\,dx\\
&\qquad + c\theta^{-1}L^2[a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2\alpha}\int_{B_3^+}\left[|Dv_j^\ell|^{2q-p}+|Dw_j|^{2q-p}\right]dx.
\end{aligned}$$

Choosing $\theta \in (0,1)$ small enough, we conclude that

$$\int_{B_3^+} |V_p(Dv_j^\ell) - V_p(Dw_j)|^2\,dx \\ \leq c[a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2\alpha}\int_{B_3^+}\left[|Dv_j^\ell|^{2q-p} + |Dw_j|^{2q-p}\right]dx \quad (4.47)$$

for some positive constant $c = c(n,p,q,\nu,L)$. We shall consider the finite difference operator $\tau_{s,h}$ for $s \in \{1,\cdots,n\}$ and $h \in \mathbb{R}$ with $|h| \in (0,10^{-4}]$, and with $h > 0$ when dealing with the case $s = n$. By a standard result on difference quotients and the estimate (4.41), we have

$$\begin{aligned}\int_{B_1^+} &|\tau_{s,h}(V_p(Dv_j^\ell))|^2\,dx \\ &\leq c(n)|h|^2\int_{B_2^+}|D(V_p(Dv_j^\ell))|^2\,dx \\ &\leq c|h|^2\int_{B_3^+}|Dv_j^\ell|^p\,dx \\ &\quad + c|h|^2\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)}|\ell|^{2(\alpha-1)} + \sigma_j\right)\int_{B_3^+}|Dv_j^\ell|^{2q-p}\,dx.\end{aligned} \quad (4.48)$$

Now we choose $\ell = h$ with $|h| \in (0,10^{-4}]$, and use (4.47) and (4.48) in order to get

$$\begin{aligned}\int_{B_1^+} &|\tau_{s,h}(V_p(Dw_j))|^2\,dx \\ &\leq c\int_{B_1^+}|\tau_{s,h}(V_p(Dv_j^h))|^2\,dx + c\int_{B_1^+}|V_p(Dv_j^h(x)) - V_p(Dw_j(x))|^2\,dx \\ &\quad + c\int_{B_1^+}|V_p(Dv_j^h(x+he_s)) - V_p(Dw_j(x+he_s))|^2\,dx \\ &\leq c\int_{B_1^+}|\tau_{s,h}(V_p(Dv_j^h))|^2\,dx + c\int_{B_3^+}|V_p(Dv_j^h) - V_p(Dw_j)|^2\,dx \\ &\leq c|h|^{2\alpha}\int_{B_3^+}|Dv_j^h|^p\,dx\end{aligned}$$

$$+ c|h|^{2\alpha}\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)} + \sigma_j\right)\int_{B_3^+}\left[|Dv_j^h|^{2q-p} + |Dw_j|^{2q-p}\right]dx. \tag{4.49}$$

Taking $\varphi = v_j^\ell - w_j \in W_0^{1,2q-p}(B_3^+)$ as a test function in (4.40), we obtain from the structural conditions of $A_j^\ell(x,\xi)$ and Young's inequality that

$$\int_{B_3^+}\left[|Dv_j^\ell|^p + a^\ell(x)|Dv_j^\ell|^q + \sigma_j|Dv_j^\ell|^{2q-p}\right]dx \leq c\int_{B_3^+}\left[|Dw_j|^p + a^\ell(x)|Dw_j|^q + \sigma_j|Dw_j|^{2q-p}\right]dx.$$

Again using Young's inequality, we get

$$\begin{aligned}\int_{B_3^+} a^\ell(x)|Dw_j|^q\,dx &\leq \|a\|_{L^\infty(B_4^+)}\int_{B_3^+}|Dw_j|^{\frac{p}{2}}|Dw_j|^{\frac{2q-p}{2}}\,dx \\ &\leq \int_{B_3^+}|Dw_j|^p\,dx + \|a\|^2_{L^\infty(B_4^+)}\int_{B_3^+}|Dw_j|^{2q-p}\,dx.\end{aligned}$$

Then it follows that

$$\int_{B_3^+}|Dv_j^\ell|^p\,dx \leq c\int_{B_3^+}|Dw_j|^p\,dx + \left(\|a\|^2_{L^\infty(B_4^+)} + \sigma_j\right)\int_{B_3^+}|Dw_j|^{2q-p}\,dx. \tag{4.50}$$

Also, comparing v_j^ℓ and w_j yields

$$\int_{B_3^+}|Dv_j^\ell|^{2q-p}\,dx \leq c\int_{B_3^+}|Dw_j|^p\,dx + c\int_{B_3^+}|Dw_j|^{2q-p}\,dx. \tag{4.51}$$

We combine the inequalities (4.49)-(4.51) with the choice $\ell = h$ to find that

$$\begin{aligned}&\int_{B_1^+}|\tau_{s,h}(V_p(Dw_j))|^2\,dx \\ &\leq c|h|^{2\alpha}\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)} + 1\right)\int_{B_4^+}|Dw_j|^p\,dx \\ &\quad + c|h|^{2\alpha}\left(\|a\|^2_{L^\infty(B_4^+)} + [a]^2_{C^{0,\alpha}(B_4^+)} + \sigma_j\right)\int_{B_4^+}|Dw_j|^{2q-p}\,dx.\end{aligned}$$

Applying the fractional Sobolev embedding theorem with [50, Lemma 2.2 and Lemma 2.3], we deduce that for every $t \in (0, \alpha)$ there exists a positive constant $c = c(n, p, q, \nu, L, t)$ such that

$$\begin{aligned}\|V_p(Dw_j)\|^2_{L^{\frac{np}{n-2t}}(B^+_{1/2})} &\le c\left(\|a\|^2_{L^\infty(B^+_4)} + [a]^2_{C^{0,\alpha}(B^+_4)} + 1\right)\|Dw_j\|^p_{L^p(B^+_4)} \\ &\quad + c\left(\|a\|^2_{L^\infty(B^+_4)} + [a]^2_{C^{0,\alpha}(B^+_4)} + \sigma_j\right)\|Dw_j\|^{2q-p}_{L^{2q-p}(B^+_4)}.\end{aligned}$$

From the covering and interpolation arguments originally introduced in the proof of [38, Theorem 5.1] and [40, Theorem 3.1], we see that for every $t \in (0, \alpha)$ and for every $\rho \in (0, 4)$, there exist an exponent $\gamma_0 = \gamma_0(n, p, q, \alpha, t) \ge 1$ and a constant $c = c(n, p, q, \alpha, \nu, L, \|a\|_{L^\infty(B^+_4)}, [a]_{C^{0,\alpha}(B^+_4)}, t, \rho) > 0$ such that

$$\|Dw_j\|_{W^{\frac{2t}{p},p}(B^+_\rho)} \le c\left[1 + \|Dw_j\|_{L^p(B^+_4)}\right]^{\gamma_0}, \qquad \text{if } p \ge 2, \tag{4.52}$$

and

$$\|Dw_j\|_{W^{t,p}(B^+_\rho)} \le c\left[1 + \|Dw_j\|_{L^p(B^+_4)}\right]^{\gamma_0}, \qquad \text{if } 1 < p < 2. \tag{4.53}$$

In addition, by the fractional Sobolev embedding theorem, we have

$$\|Dw_j\|_{L^{\frac{np}{n-2t}}(B^+_\rho)} \le c\left[1 + \|Dw_j\|_{L^p(B^+_4)}\right]^{\gamma_0}. \tag{4.54}$$

Using a standard diagonal argument as in the proof of [40, Theorem 3.1], we see that there exist a subsequence, which we still denote by $\{w_j\}_{j=1}^\infty$, such that $Dw_j \longrightarrow Dw$ in $L^p_{\mathrm{loc}}(B^+_4)$ and a.e. in B^+_4. Moreover, it follows from (4.52), (4.53) and the fractional Sobolev embedding theorem that $Dw_j \longrightarrow Dw$ in $L^{\frac{np}{n-2t}}_{\mathrm{loc}}(B^+_4)$ for every $t \in (0, \alpha)$. Letting $k \to \infty$ in (4.52)-(4.54), we obtain the local regularity result (4.24). Now, from the Remark 4.1.11 below, we have $Dw_j \longrightarrow Dw$ in $L^{2q-p}_{\mathrm{loc}}(B^+_4)$. Getting $j \to \infty$ in the weak formulation of (4.31), we conclude that $w \in u + W^{1,p}_0(B^+_4)$ is a distributional solution to $\operatorname{div} A(x, Dw) = 0$ in B^+_4 with $H(x, Dw) \in L^1(B^+_4)$.

We next show the uniqueness of solutions. Suppose that w_1, w_2 are distributional solutions to (4.22) with $r = 1$ and $H(x, Dw_1), H(x, Dw_2) \in L^1(B^+_4)$. In light of Proposition 4.1.7, we can take $\varphi = w_1 - w_2 \in W^{1,1}_0(B^+_4)$ as a test function, and hence

$$\int_{B_4^+} \langle A(x, Dw_1) - A(x, Dw_2), Dw_1 - Dw_2 \rangle \, dx = 0.$$

From the monotonicity property of $A(x, \xi)$, we get $D(w_1 - w_2) = 0$ in B_4^+. Since $w_1 - w_2 = 0$ on ∂B_4^+, we conclude that $w_1 - w_2 = 0$ in B_4^+. $\square$

We also introduce a boundary version of the conditional reverse Hölder type inequality. For the interior case, we refer the reader to [38, 40].

Lemma 4.1.10. *Under the assumptions (4.2) and (1.13), consider a function $u \in W^{1,1}(B_{5r}^+)$ with $0 < r \leq 1$, $H(x, Du) \in L^1(B_{5r}^+)$ and*

$$u = 0 \quad \text{on} \quad T_{5r}.$$

Let $w \in W^{1,p}(B_{4r}^+)$ be the distributional solution to (4.22) with $H(x, Dw) \in L^1(B_{4r}^+)$. If the inequality

$$\sup_{x \in B_{4r}^+} a(x) \leq K[a]_{C^{0,\alpha}(B_{4r}^+)} r^\alpha \tag{4.55}$$

holds for some $K \geq 1$, then for every $\widetilde{q} < \frac{np}{n-2\alpha}$, the following reverse Hölder type inequality

$$\left(\fint_{B_{3r}^+} |Dw|^{\widetilde{q}} \, dx \right)^{1/\widetilde{q}} \leq c \left(\fint_{B_{4r}^+} |Dw|^p \, dx \right)^{1/p} \tag{4.56}$$

holds for some positive constant $c = c(\mathtt{data}, \widetilde{q}, K)$.

Proof. We first notice that it is sufficient to prove (4.56) in the case

$$\widetilde{q} = \frac{np}{n - 2t} \qquad \text{for} \quad t \in \left(n\left(\frac{q}{p} - 1\right), \alpha \right).$$

From the proofs of the previous lemma and [38, Theorem 5.1], we deduce that for every $t \in (n(q/p - 1), \alpha)$ there exists a positive constant $c =$

$c(n,p,q,\nu,L,t)$ such that the inequality

$$
\begin{aligned}
&\left(\fint_{B_{3r}^+} |Dw|^{\frac{np}{n-2t}}\,dx\right)^{\frac{n-2t}{n}} \\
&\quad\le c\left(1+\|a\|^2_{L^\infty(B_{4r}^+)} + r^{2\alpha}[a]^2_{C^{0,\alpha}(B_{4r}^+)}\right)\fint_{B_{4r}^+}|Dw|^p\,dx \\
&\qquad + c\left(\|a\|^2_{L^\infty(B_{4r}^+)} + r^{2\alpha}[a]^2_{C^{0,\alpha}(B_{4r}^+)}\right)^{b_1}\left(\fint_{B_{4r}^+}|Dw|^p\,dx\right)^{b_2}
\end{aligned}
$$

holds for exponents

$$
b_1 = \frac{tp}{tp-n(q-p)} \ge 1, \quad b_2 = \frac{t(2q-p)-n(q-p)}{tp-n(q-p)} \ge 1. \tag{4.57}
$$

Then it follows from (4.55) that

$$
\left(\fint_{B_{3r}^+} |Dw|^{\frac{np}{n-2t}}\,dx\right)^{\frac{n-2t}{np}} \le cM\left(\fint_{B_{4r}^+}|Dw|^p\,dx\right)^{\frac{1}{p}}, \tag{4.58}
$$

where $c = c(n,p,q,\nu,L,t) > 0$ and

$$
M := \left[1+(K^2+1)[a]^2_{C^{0,\alpha}}r^{2\alpha} + (K^2+1)^{b_1}[a]^{2b_1}_{C^{0,\alpha}}r^{2\alpha b_1 - n(b_2-1)}\left(\int_{B_{4r}^+}|Dw|^p\,dx\right)^{b_2-1}\right]^{\frac{1}{p}}.
$$

We see from (4.57) and (1.13) that

$$
2\alpha b_1 - n(b_2-1) = \frac{2tpn}{tp-n(q-p)}\left(1+\frac{\alpha}{n}-\frac{q}{p}\right) > 0.
$$

Since $r \le 1$, we have

$$
M \le 1 + (K^2+1)^{\frac{1}{p}}[a]^{\frac{2}{p}}_{C^{0,\alpha}} + (K^2+1)^{\frac{b_1}{p}}[a]^{\frac{2b_1}{p}}_{C^{0,\alpha}}\|Dw\|^{b_2-1}_{L^p(B_{4r}^+)} \le c(\mathtt{data},t,K).
$$

This inequality and (4.58) yield the desired inequality (4.56). $\square$

Remark 4.1.11. *From (1.13), we have*

$$q < \left(1 + \frac{\alpha}{n}\right) p < \left(1 + \frac{2\alpha}{n - 2\alpha}\right) p = \frac{np}{n - 2\alpha}. \tag{4.59}$$

Hence we can take $\widetilde{q} = q$ in Lemma 4.1.10. As a matter of fact, (4.25) in Lemma 4.1.9 is derived from (4.24) and (4.59).

The remainder of this subsection will be devoted to Lipschitz estimates for reference problems in which the nonlinearity has no x-dependence. To be specific, consider a vector-valued function $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following structural conditions:

$$|A_0(\xi)| + |\xi||D_\xi A_0(\xi)| \leq L \left(|\xi|^{p-1} + a_0|\xi|^{q-1}\right), \tag{4.60}$$

$$\nu \left(|\xi|^{p-2} + a_0|\xi|^{q-2}\right) |\eta|^2 \leq \langle D_\xi A_0(\xi)\eta, \eta\rangle, \tag{4.61}$$

for every $\xi, \eta \in \mathbb{R}^n$, where $0 < \nu \leq L < +\infty$, $1 < p < q$ and $a_0 \geq 0$ are fixed constants. As in (4.11), we shall use the notation

$$H_0(\xi) := |\xi|^p + a_0|\xi|^q, \tag{4.62}$$

for $\xi \in \mathbb{R}^n$.

Then we have the following Lipschitz estimate.

Lemma 4.1.12. *[56, 88] Let $v \in W^{1,1}(B_{2r})$ be a distributional solution to*

$$\text{div } A_0(Dv) = 0 \quad \text{in} \quad B_{2r}, \tag{4.63}$$

with $H_0(Dv) \in L^1(B_{2r})$. Then we have $Dv \in L^\infty(B_r)$ with the estimate

$$\sup_{B_r} H_0(Dv) \leq c \fint_{B_{2r}} H_0(Dv)\, dx, \tag{4.64}$$

where $c = (n, p, q, \nu, L)$ is a universal constant.

A boundary version of the above Lemma is the following:

Lemma 4.1.13. *[40] Let $v \in W^{1,1}(B_{2r}^+)$ be a distributional solution to*

$$\begin{cases} \text{div } A_0(Dv) &= 0 \quad \text{in} \quad B_{2r}^+, \\ v &= 0 \quad \text{on} \quad T_{2r}, \end{cases} \tag{4.65}$$

with $H_0(Dv) \in L^1(B_{2r}^+)$. Then we have $Dv \in L^\infty(B_r^+)$ with the estimate

$$\sup_{B_r^+} H_0(Dv) \le c \fint_{B_{2r}^+} H_0(Dv)\,dx, \tag{4.66}$$

where $c = (n, p, q, \nu, L)$ is a universal constant.

4.1.3 Comparison estimates

In this subsection we focus on $B_R^+ = B_R^+(0) \cap \{x_n > 0\}$ with $R \le \widetilde{R}$, where $\widetilde{R} = \widetilde{R}(\texttt{data}, \gamma) > 0$ will be determined later. But we first assume that $\widetilde{R} \le 1$.

We start with a covering argument. Let $u \in W^{1,1}(B_R^+)$ be a distributional solution to

$$\begin{cases} \operatorname{div} A(x, Du) &= \operatorname{div} G(x, F) \quad \text{in} \quad B_R^+, \\ u &= 0 \quad \text{on} \quad T_R, \end{cases} \tag{4.67}$$

with

$$H(x, Du),\ H(x, F) \in L^1(B_R^+). \tag{4.68}$$

The nonlinearity $A : B_R^+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (4.5)-(4.7) with the assumptions (4.2) and (1.13). We first select radii r_1, r_2 such that $\frac{R}{2} \le r_1 < r_2 \le R$ and consider the upper level sets

$$E^+(\lambda, s) := \left\{x \in B_s^+ : H(x, Du(x)) > \lambda\right\}, \quad \frac{R}{2} \le s \le R, \quad \lambda > 0. \tag{4.69}$$

For a fixed point $y \in E^+(\lambda, r_1)$, we define a continuous function $\Phi_y : (0, r_2 - r_1] \to [0, \infty)$ by

$$\Phi_y(\rho) := \fint_{B_\rho^+(y)} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx, \tag{4.70}$$

with $B_\rho^+(y) \equiv B_\rho(y) \cap \{x_n > 0\}$, where $\delta > 0$ is to be determined later. From the Lebesgue differentiation theorem and (4.69), it follows that for almost every $y \in E^+(\lambda, r_1)$,

$$\begin{aligned} \lim_{\rho \to 0} \Phi_y(\rho) &= H(y, Du(y)) + \frac{1}{\delta} H(y, F(y)) \\ &\ge H(y, Du(y)) > \lambda. \end{aligned} \tag{4.71}$$

On the other hand, for any $\rho \in \left[\frac{r_2-r_1}{100}, r_2 - r_1\right]$, we obtain

$$\begin{aligned}
\Phi_y(\rho) &= \fint_{B_\rho^+(y)} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx \\
&\le \frac{|B_{r_2}^+|}{|B_\rho^+(y)|} \fint_{B_{r_2}^+} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx \\
&\le \left(\frac{r_2}{\rho}\right)^n \fint_{B_{r_2}^+} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx \\
&\le \left(\frac{r_2}{\frac{r_2-r_1}{100}}\right)^n \fint_{B_{r_2}^+} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx \\
&= \frac{100^n r_2^n}{(r_2-r_1)^n} \fint_{B_{r_2}^+} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx =: \lambda_0. \qquad (4.72)
\end{aligned}$$

From now on, we consider positive numbers λ satisfying

$$\lambda > \lambda_0. \tag{4.73}$$

Since Φ_y is continuous, it follows from (4.71)-(4.73) that for almost every $y \in E^+(\lambda, r_1)$, there exists an exit time radius $\rho_y \in \left(0, \frac{r_2-r_1}{100}\right)$ such that

$$\Phi_y(\rho_y) = \lambda \quad \text{and} \quad \Phi_y(\rho) < \lambda \quad \text{for all } \rho \in (\rho_y, r_2 - r_1]. \tag{4.74}$$

Note that the family $\{B_{\rho_y}^+(y)\}$ covers $E^+(\lambda, r_1)$ up to a negligible set. Applying Vitali's covering lemma, there exists a countable family of disjoint sets $\{B_{\rho_i}^+(y_i)\}_{i=1}^\infty$ with $y_i \in E^+(\lambda, r_1)$ and $\rho_i = \rho_{y_i} \in \left(0, \frac{r_2-r_1}{100}\right)$ such that

$$E^+(\lambda, r_1) \subset \bigcup_{i\ge1} B_{5\rho_i}^+(y_i) \cup \text{negligible set}, \tag{4.75}$$

and

$$\Phi_{y_i}(\rho_i) = \lambda \quad \text{and} \quad \Phi_{y_i}(\rho) < \lambda \quad \text{for all } \rho \in (\rho_i, r_2 - r_1]. \tag{4.76}$$

The comparison estimates will be divided into two cases depending on whether $B_{15\rho_i}(y_i) \subset B_R^+$ or $B_{15\rho_i}(y_i) \not\subset B_R^+$. First, for the interior case

$B_{15\rho_i}(y_i) \subset B_R^+$, we set

$$B_i^0 \equiv B_{\rho_i}(y_i), \quad B_i^j \equiv B_{5j\rho_i}(y_i) \ \text{ for } \ j = 1, 2, 3. \tag{4.77}$$

We recall that

$$0 < \rho_i < 5j\rho_i \leq 15\rho_i < 100\rho_i < r_2 - r_1 \leq \frac{R}{2} \leq \frac{\widetilde{R}}{2} < \widetilde{R} \leq 1 \tag{4.78}$$

and hence (4.70) and (4.76) yield, for $j = 0, 1, 2, 3$,

$$\fint_{B_i^j} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx < \lambda. \tag{4.79}$$

Next, for the boundary case $B_{15\rho_i}(y_i) \not\subset B_R^+$, we fix a boundary point $\widehat{y_i} \in B_{15\rho_i}(y_i) \cap T_R$. Since $|y_i - \widehat{y_i}| < 15\rho_i$, we see that

$$B^+_{5\rho_i}(y_i) \subset B^+_{20\rho_i}(\widehat{y_i}) \subset B^+_{80\rho_i}(\widehat{y_i}) \subset B^+_{100\rho_i}(y_i). \tag{4.80}$$

Now we set

$$B_i^{0+} \equiv B^+_{\rho_i}(\widehat{y_i}), \quad B_i^{j+} \equiv B^+_{20j\rho_i}(\widehat{y_i}) \ \text{ for } \ j = 1, 2, 3, 4. \tag{4.81}$$

Then we deduce that, for $j = 1, 2, 3, 4$,

$$\begin{aligned} &\fint_{B_i^{j+}} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\ &\quad \leq \frac{|B^+_{100\rho_i}(y_i)|}{|B^+_{20\rho_i}(\widehat{y_i})|} \fint_{B^+_{100\rho_i}(y_i)} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\ &\quad \leq 2 \cdot 5^n \fint_{B^+_{100\rho_i}(y_i)} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\ &\quad < 2 \cdot 5^n \lambda. \end{aligned} \tag{4.82}$$

Consequently, we have

$$\fint_{B_i^{4+}} H(x, Du)\, dx \leq c\lambda \quad \text{and} \quad \fint_{B_i^{4+}} H(x, F)\, dx \leq c\delta\lambda. \tag{4.83}$$

The comparison estimates for the interior case can be proved in a similar way as those for the boundary one. For this reason, we mainly deal with the boundary case. We consider now the first reference problem

$$\begin{cases} \operatorname{div} A(x, Dw_i) &= 0 \quad \text{in} \quad B_i^{4+}, \\ w_i &= u \quad \text{on} \quad \partial B_i^{4+}. \end{cases} \tag{4.84}$$

According to Lemma 4.1.9, there exists a unique distributional solution w_i to the above problem (4.84) with $H(x, Dw_i) \in L^1(B_i^{4+})$. From the local regularity results (4.24) and (4.25) in Lemma 4.1.9, we have

$$Dw_i \in L^{\frac{np}{n-2t}}(B_i^{3+}) \ \text{ for every } t < \alpha, \tag{4.85}$$

and

$$Dw_i \in L^q(B_i^{3+}). \tag{4.86}$$

In addition, it follows from (4.23) and (4.84) that

$$\fint_{B_i^{4+}} H(x, Dw_i)\,dx \le c \fint_{B_i^{4+}} H(x, Du)\,dx \le c\lambda, \tag{4.87}$$

holds for a constant $c = c(n, p, q, \nu, L)$.

We can now prove a first comparison result.

Lemma 4.1.14. *Let $u \in W^{1,1}(B_R^+)$ be a distributional solution to (4.67) with (4.68), under the assumptions (4.2) and (1.13). Then for any $0 < \varepsilon < 1$, there exists a constant $\delta = \delta(n, p, q, \nu, L, \varepsilon) > 0$ such that if (4.83) holds and if $w_i \in W^{1,1}(B_i^{4+})$ is the distributional solution to (4.84) with $H(x, Dw_i) \in L^1(B_i^{4+})$, then we have*

$$\fint_{B_i^{4+}} H(x, Du - Dw_i)\,dx \le \varepsilon\lambda. \tag{4.88}$$

Proof. We take $\varphi = u - w_i \in W_0^{1,1}(B_i^{4+})$ as a test function in (4.67) and (4.84) to find that

$$\fint_{B_i^{4+}} \langle A(x, Du) - A(x, Dw_i), D(u - w_i)\rangle\,dx = \fint_{B_i^{4+}} \langle G(x, F), D(u - w_i)\rangle\,dx. \tag{4.89}$$

Indeed, Proposition 4.1.7 ensures that the above choice of φ is valid from

(4.87).
If $2 \le p < q$, (4.10) implies

$$\fint_{B_i^{4+}} H(x, Du - Dw_i)\,dx \le c \fint_{B_i^{4+}} \langle A(x, Du) - A(x, Dw_i), D(u - w_i)\rangle\,dx. \tag{4.90}$$

If $1 < p < q \le 2$, (4.9) implies

$$\begin{aligned}\fint_{B_i^{4+}} &\left((|Du| + |Dw_i|)^{p-2} + a(x)(|Du| + |Dw_i|)^{q-2}\right)|Du - Dw_i|^2\,dx \\ &\le c \fint_{B_i^{4+}} \langle A(x, Du) - A(x, Dw_i), D(u - w_i)\rangle\,dx. \end{aligned}\tag{4.91}$$

Using Young's inequality with $\theta \in (0, 1)$, we have

$$\begin{aligned}
&\fint_{B_i^{4+}} H(x, Du - Dw_i)\,dx = \fint_{B_i^{4+}} (|Du - Dw_i|^p + a(x)|Du - Dw_i|^q)\,dx \\
&= \fint_{B_i^{4+}} (|Du| + |Dw_i|)^{\frac{p(2-p)}{2}} \left[(|Du| + |Dw_i|)^{\frac{p(p-2)}{2}} |Du - Dw_i|^p\right] dx \\
&\quad + \fint_{B_i^{4+}} a(x)(|Du| + |Dw_i|)^{\frac{q(2-q)}{2}} \left[(|Du| + |Dw_i|)^{\frac{q(q-2)}{2}} |Du - Dw_i|^q\right] dx \\
&\le \theta \fint_{B_i^{4+}} (|Du| + |Dw_i|)^p\,dx + c\theta^{-\frac{2-p}{p}} \fint_{B_i^{4+}} (|Du| + |Dw_i|)^{p-2} |Du - Dw_i|^2\,dx \\
&\quad + \theta \fint_{B_i^{4+}} a(x)(|Du| + |Dw_i|)^q\,dx \\
&\quad + c\theta^{-\frac{2-q}{q}} \fint_{B_i^{4+}} a(x)(|Du| + |Dw_i|)^{q-2} |Du - Dw_i|^2\,dx \\
&\le 2^{p-1}\theta \fint_{B_i^{4+}} (|Du|^p + |Dw_i|^p)\,dx \\
&\quad + c\theta^{-\frac{2-p}{p}} \fint_{B_i^{4+}} (|Du| + |Dw_i|)^{p-2} |Du - Dw_i|^2\,dx \\
&\quad + 2^{q-1}\theta \fint_{B_i^{4+}} a(x)(|Du|^q + |Dw_i|^q)\,dx \\
&\quad + c\theta^{-\frac{2-q}{q}} \fint_{B_i^{4+}} a(x)(|Du| + |Dw_i|)^{q-2} |Du - Dw_i|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le c\theta \fint_{B_i^{4+}} [H(x, Du) + H(x, Dw_i)]\, dx \\
&\quad + \frac{c}{\theta} \fint_{B_i^{4+}} \big((|Du| + |Dw_i|)^{p-2} + a(x)(|Du| + |Dw_i|)^{q-2}\big)\, |Du - Dw_i|^2\, dx,
\end{aligned} \tag{4.92}
$$

since $0 < \theta < 1$ and $1 \le \theta^{-\frac{2-q}{q}} < \theta^{-\frac{2-p}{p}} < \theta^{-1}$. From (4.87), (4.91) and (4.92), it follows that

$$
\begin{aligned}
&\fint_{B_i^{4+}} H(x, Du - Dw_i)\, dx \\
&\qquad \le c\theta\lambda + \frac{c}{\theta} \fint_{B_i^{4+}} \langle A(x, Du) - A(x, Dw_i), D(u - w_i)\rangle\, dx.
\end{aligned} \tag{4.93}
$$

Likewise, the same conclusion can be drawn for the case $1 < p < 2 < q$.

We next estimate the right-hand side of (4.89). We use the growth condition (4.8), Young's inequality with $\tau \in (0, 1)$ and (4.83), to discover

$$
\begin{aligned}
&\fint_{B_i^{4+}} \langle G(x, F), D(u - w_i)\rangle\, dx \\
&\quad \le L \fint_{B_i^{4+}} (|F|^{p-1} + a(x)|F|^{q-1})|Du - Dw_i|\, dx \\
&\quad \le L \left[\tau \fint_{B_i^{4+}} H(x, Du - Dw_i)\, dx + c\tau^{-\frac{1}{p-1}} \fint_{B_i^{4+}} H(x, F)\, dx \right] \\
&\quad \le c\tau \fint_{B_i^{4+}} H(x, Du - Dw_i)\, dx + c\tau^{-\frac{1}{p-1}} \delta\lambda.
\end{aligned} \tag{4.94}
$$

Combining (4.89), (4.90), (4.93) and (4.94), we find that

$$
\begin{aligned}
&\fint_{B_i^{4+}} H(x, Du - Dw_i)\, dx \\
&\qquad \le c_0\theta\lambda + c_0 \frac{\tau}{\theta} \fint_{B_i^{4+}} H(x, Du - Dw_i)\, dx + c_0 \frac{\tau^{-\frac{1}{p-1}}}{\theta} \delta\lambda,
\end{aligned} \tag{4.95}
$$

for some universal constant $c_0 > 1$ depending only on n, p, q, ν, L. But then

(4.88) follows by taking

$$\theta = \frac{\varepsilon}{4c_0}, \quad \tau = \frac{\varepsilon}{8c_0^2} \quad \text{and} \quad \delta = \left(\frac{\varepsilon}{4c_0}\right)^2 \left(\frac{\varepsilon}{8c_0^2}\right)^{\frac{1}{p-1}}. \tag{4.96}$$

This finishes the proof. $\square$

We next introduce a second reference problem. Let $x_{i,M} \in \overline{B_i^{3+}}$ be a point such that

$$a(x_{i,M}) = \sup_{x \in B_i^{3+}} a(x). \tag{4.97}$$

We note that (4.86) gives

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i)\, dx < +\infty. \tag{4.98}$$

By (4.98) and Lemma 4.1.9 with the choice of $a(\cdot) \equiv a(x_{i,M})$, there exists a unique distributional solution v_i to the second reference problem

$$\begin{cases} \operatorname{div} A(x_{i,M}, Dv_i) &= 0 \quad \text{in} \quad B_i^{3+}, \\ v_i &= w_i \quad \text{on} \quad \partial B_i^{3+}. \end{cases} \tag{4.99}$$

From the energy estimate (4.23) with the choice of $a(\cdot) \equiv a(x_{i,M})$, we have

$$\fint_{B_i^{3+}} H(x_{i,M}, Dv_i)\, dx \le c \fint_{B_i^{3+}} H(x_{i,M}, Dw_i)\, dx, \tag{4.100}$$

for a constant $c = c(n, p, q, \nu, L)$.

We want to find comparison estimates for v_i and w_i, as in Lemma 4.1.14 for w_i and u. To this end, we will consider the two alternatives:

$$a(x_{i,M}) = \sup_{x \in B_i^{3+}} a(x) > K[a]_{C^{0,\alpha}} \rho_i^\alpha, \tag{4.101}$$

which is called *the* (p,q)*-phase*, and

$$a(x_{i,M}) = \sup_{x \in B_i^{3+}} a(x) \le K[a]_{C^{0,\alpha}} \rho_i^\alpha, \tag{4.102}$$

which is called *the* p*-phase*, where $K = K(\mathtt{data}, \gamma) > 240$ will be determined later. In the case (4.101) occurs, the q-growth term in the energy becomes the main term. On the other hand, in the case (4.102) occurs, the q-growth term can be controlled by the p-growth one.

We now have a second comparison result as follows.

Lemma 4.1.15. *Let $u \in W^{1,1}(B_R^+)$ be a distributional solution to (4.67) with (4.68), under the assumptions (4.2) and (1.13). Let (4.83) hold and let $w_i \in W^{1,1}(B_i^{4+})$ be the distributional solution to (4.84) with $H(x, Dw_i) \in L^1(B_i^{4+})$ and $v_i \in W^{1,1}(B_i^{3+})$ be the distributional solution to (4.99) with $H(x_{i,M}, Dv_i) \in L^1(B_i^{3+})$.*

(i) If (4.101) holds, then we have

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx \le \frac{c\lambda}{K^{\frac{p-1}{2p-1}}} \tag{4.103}$$

and

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i)\, dx \le c\lambda, \tag{4.104}$$

where $c = c(n, p, q, \nu, L)$ is a universal constant.

(ii) If (4.102) holds, then we have

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx \le c(K)\rho_i^{\sigma_0}\lambda \tag{4.105}$$

and

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i)\, dx \le c(K)\lambda, \tag{4.106}$$

where $c(K) \equiv c(\mathtt{data}, K)$ is a positive constant and

$$\sigma_0 = \left(\frac{p-1}{2p-1}\right)\left[\alpha - n\left(\frac{q}{p} - 1\right)\right] > 0.$$

Proof. We take $\varphi = w_i - v_i \in W_0^{1,1}(B_i^{3+})$ as a test function in (4.84) and (4.99), since Proposition 4.1.7 ensures that the above choice of φ is valid from (4.97), (4.98) and (4.100), to discover that

$$\begin{aligned}
&\fint_{B_i^{3+}} \langle A(x_{i,M}, Dv_i) - A(x_{i,M}, Dw_i), D(v_i - w_i)\rangle \, dx \\
&\qquad = \fint_{B_i^{3+}} \langle A(x, Dw_i) - A(x_{i,M}, Dw_i), D(v_i - w_i)\rangle \, dx. \qquad (4.107)
\end{aligned}$$

A careful analysis similar to that in the proof of Lemma 4.1.14 shows that for any $\theta \in (0,1)$,

$$\begin{aligned}
&\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i) \, dx \\
&\qquad \le c\theta\lambda + \frac{c}{\theta} \fint_{B_i^{3+}} \langle A(x_{i,M}, Dv_i) - A(x_{i,M}, Dw_i), D(v_i - w_i)\rangle \, dx. \qquad (4.108)
\end{aligned}$$

On the other hand, it follows from (4.81) and (4.97) that

$$\begin{aligned}
&\fint_{B_i^{3+}} \langle A(x, Dw_i) - A(x_{i,M}, Dw_i), D(v_i - w_i)\rangle \, dx \\
&\qquad \le \fint_{B_i^{3+}} |A(x, Dw_i) - A(x_{i,M}, Dw_i)||Dw_i - Dv_i| \, dx \\
&\qquad \le L \fint_{B_i^{3+}} |a(x) - a(x_{i,M})||Dw_i|^{q-1}|Dw_i - Dv_i| \, dx \\
&\qquad \le L \left(\operatorname*{osc}_{B_i^{3+}} a \right) \fint_{B_i^{3+}} |Dw_i|^{q-1}|Dw_i - Dv_i| \, dx. \qquad (4.109)
\end{aligned}$$

Combining (4.107)-(4.109), we obtain

$$\begin{aligned}
&\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i) \, dx \\
&\qquad \le c\theta\lambda + \frac{c}{\theta} \left(\operatorname*{osc}_{B_i^{3+}} a \right) \fint_{B_i^{3+}} |Dw_i|^{q-1}|Dw_i - Dv_i| \, dx, \qquad (4.110)
\end{aligned}$$

for some constant $c = c(n, p, q, \nu, L)$. A task is now to estimate the right-hand side of (4.110).

(i) First, we consider the case when (4.101) holds. Note that

$$\inf_{B_i^{3+}} a = \sup_{B_i^{3+}} a - \operatorname*{osc}_{B_i^{3+}} a \geq K[a]_{C^{0,\alpha}}\rho_i^\alpha - 120[a]_{C^{0,\alpha}}\rho_i^\alpha = (K-120)[a]_{C^{0,\alpha}}\rho_i^\alpha,$$

and hence

$$\operatorname*{osc}_{B_i^{3+}} a \leq 120[a]_{C^{0,\alpha}}\rho_i^\alpha \leq \frac{120}{K-120}a(x) \leq \frac{240}{K}a(x), \quad \forall x \in B_i^{3+}, \tag{4.111}$$

since $K > 240$. Furthermore, it follows that

$$a(x_{i,M}) \leq a(x) + 120[a]_{C^{0,\alpha}}\rho_i^\alpha \leq a(x) + \frac{240}{K}a(x) \leq 2a(x), \quad \forall x \in B_i^{3+}. \tag{4.112}$$

From (4.112) and (4.87), we obtain

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i)\,dx \leq 2\fint_{B_i^{3+}} H(x, Dw_i)\,dx \leq c\lambda, \tag{4.113}$$

which gives (4.104).

Now, in order to get (4.103), let us estimate (4.110). Using (4.111), (4.87) and Young's inequality with $\tau \in (0,1)$, we obtain

$$\begin{aligned}
&\operatorname*{osc}_{B_i^{3+}} a \fint_{B_i^{3+}} |Dw_i|^{q-1}|Dw_i - Dv_i|\,dx \\
&\quad \leq \frac{c}{K}\fint_{B_i^{3+}} a(x)|Dw_i|^{q-1}|Dw_i - Dv_i|\,dx \\
&\quad \leq \frac{c}{K}\left[\tau^{-\frac{1}{q-1}}\fint_{B_i^{3+}} a(x)|Dw_i|^q\,dx + \tau\fint_{B_i^{3+}} a(x)|Dw_i - Dv_i|^q\,dx\right] \\
&\quad \leq \frac{c}{K}\left[\tau^{-\frac{1}{q-1}}\fint_{B_i^{3+}} H(x, Dw_i)\,dx + \tau\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\,dx\right] \\
&\quad \leq \frac{c}{K}\left[\tau^{-\frac{1}{q-1}}\lambda + \tau\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\,dx\right] \\
&\quad \leq \frac{c\tau^{-\frac{1}{p-1}}\lambda}{K} + c\tau\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\,dx.
\end{aligned} \tag{4.114}$$

The last inequality is due to the facts that $1 < p < q$ and $K > 240$. Combining (4.110) with (4.114), we have

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx \le c\theta\lambda + \frac{c\tau^{-\frac{1}{p-1}}\lambda}{K\theta} + \frac{c\tau}{\theta} \fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx.$$

Taking $\tau = \dfrac{\theta}{2c}$ in the above inequality, we get

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx \le c\theta\lambda + \frac{c}{K\theta^{\frac{p}{p-1}}}\lambda,$$

for a universal constant $c = c(n, p, q, \nu, L)$. Now the conclusion (4.103) follows by taking $\theta = \dfrac{1}{K^{\frac{p-1}{2p-1}}} \in (0, 1)$.

(ii) We now turn to the case when (4.102) holds. Since $a(\cdot) \ge 0$, it is clear that $\operatorname{osc}_{B_i^{3+}} a \le \sup_{B_i^{3+}} a = a(x_{i,M})$. Using this fact and Young's inequality with $\tau \in (0, 1)$, we obtain

$$\begin{aligned}
\operatorname*{osc}_{B_i^{3+}} a \fint_{B_i^{3+}} |Dw_i|^{q-1}|Dw_i - Dv_i|\, dx &\le \fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^{q-1}|Dw_i - Dv_i|\, dx \\
&\le \tau^{-\frac{1}{q-1}} \fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^q\, dx + \tau \fint_{B_i^{3+}} a(x_{i,M})|Dw_i - Dv_i|^q\, dx \\
&\le \tau^{-\frac{1}{p-1}} \fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^q\, dx + \tau \fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx.
\end{aligned} \tag{4.115}$$

Combining (4.110) with (4.115), we have

$$\begin{aligned}
\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx \le c\theta\lambda &+ \frac{c\tau^{-\frac{1}{p-1}}}{\theta} \fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^q\, dx \\
&+ \frac{c\tau}{\theta} \fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\, dx.
\end{aligned}$$

Taking $\tau = \dfrac{\theta}{2c}$ in the above inequality, we get

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\,dx \le c\theta\lambda + \frac{c}{\theta^{\frac{p}{p-1}}} \fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^q\,dx, \quad (4.116)$$

for some constant $c = c(n, p, q, \nu, L)$. We now employ (4.20), (4.87), (4.102) and Lemma 4.1.10 with $\widetilde{q} = q$ to estimate the integrals in the right-hand side of (4.116) as follows:

$$\begin{aligned}
&\fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^q\,dx \\
&\qquad \le c\rho_i^\alpha \fint_{B_i^{3+}} |Dw_i|^q\,dx \le c\rho_i^\alpha \left(\fint_{B_i^{4+}} |Dw_i|^p\,dx \right)^{\frac{q}{p}} \\
&\qquad \le c\rho_i^{\alpha - n\left(\frac{q}{p}-1\right)} \left(\int_{B_i^{4+}} |Dw_i|^p\,dx \right)^{\frac{q}{p}-1} \fint_{B_i^{4+}} |Dw_i|^p\,dx \\
&\qquad \le c\rho_i^{\alpha - n\left(\frac{q}{p}-1\right)} \left(\int_{B_i^{4+}} H(x, Dw_i)\,dx \right)^{\frac{q}{p}-1} \fint_{B_i^{4+}} H(x, Dw_i)\,dx \\
&\qquad \le c\rho_i^{\alpha - n\left(\frac{q}{p}-1\right)} \left(\int_{B_i^{4+}} H(x, Du)\,dx \right)^{\frac{q}{p}-1} \fint_{B_i^{4+}} H(x, Dw_i)\,dx \\
&\qquad \le c\rho_i^{\alpha - n\left(\frac{q}{p}-1\right)} \|H(\cdot, Du)\|_{L^1(B_R^+)}^{\frac{q}{p}-1} \lambda \\
&\qquad \le c\rho_i^{\alpha - n\left(\frac{q}{p}-1\right)} \|H(\cdot, F)\|_{L^1(B_R^+)}^{\frac{q}{p}-1} \lambda \\
&\qquad \le c\rho_i^{\alpha - n\left(\frac{q}{p}-1\right)} \lambda, \qquad (4.117)
\end{aligned}$$

for some constant $c = c(\mathtt{data}, K)$. But then since $\rho_i < 1$ and $\dfrac{q}{p} < 1 + \dfrac{\alpha}{n}$, one can rewrite (4.117) as

$$\fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^q\,dx \le c\lambda. \quad (4.118)$$

This estimate, (4.87) and (4.118) imply that

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i)\,dx \le \fint_{B_i^{3+}} H(x, Dw_i)\,dx + \fint_{B_i^{3+}} a(x_{i,M})|Dw_i|^q\,dx \le c\lambda. \tag{4.119}$$

Therefore, we deduce from (4.116), (4.117) and (4.119) that

$$\fint_{B_i^{3+}} H(x_{i,M}, Dw_i - Dv_i)\,dx \le c\theta\lambda + \frac{c\rho_i^{\alpha-n\left(\frac{q}{p}-1\right)}\lambda}{\theta^{\frac{p}{p-1}}}, \tag{4.120}$$

for a constant $c = c(\texttt{data}, K)$. Now the conclusion (4.105) follows by taking

$$\theta = \rho_i^{\left(\frac{p-1}{2p-1}\right)\left[\alpha-n\left(\frac{q}{p}-1\right)\right]} = \rho_i^{\sigma_0} \in (0,1).$$

Note that we have also obtained (4.106) from (4.119). This completes the proof. $\square$

The following lemma provides a Lipschitz estimate for v_i.

Lemma 4.1.16. *Let $u \in W^{1,1}(B_R^+)$ be a distributional solution to (4.67) with (4.68), under the assumptions (4.2) and (1.13). Suppose that (4.83) hold. Let $w_i \in W^{1,1}(B_i^{4+})$ be the distributional solution to (4.84) with $H(x, Dw_i) \in L^1(B_i^{4+})$, and let $v_i \in W^{1,1}(B_i^{3+})$ be the distributional solution to (4.99) with $H(x_{i,M}, Dv_i) \in L^1(B_i^{3+})$. Then we have*

$$\sup_{B_i^{1+}} H(x_{i,M}, Dv_i) \le c\lambda \tag{4.121}$$

for some positive constant $c = c(\texttt{data})$.

Proof. The proof is based on Lemma 4.1.13 and Lemma 4.1.15. We first claim that the energy estimate

$$\fint_{B_i^{3+}} H(x_{i,M}, Dv_i)\,dx \le c\fint_{B_i^{3+}} H(x_{i,M}, Dw_i)\,dx \le c\lambda \tag{4.122}$$

holds for some positive constant $c = c(\texttt{data})$. To this end, consider the two alternatives:

$$a(x_{i,M}) = \sup_{x\in B_i^{3+}} a(x) > 10^3[a]_{C^{0,\alpha}}\rho_i^\alpha \tag{4.123}$$

and

$$a(x_{i,M}) = \sup_{x \in B_i^{3+}} a(x) \leq 10^3 [a]_{C^{0,\alpha}} \rho_i^\alpha, \tag{4.124}$$

which are (4.101) and (4.102) for $K = 10^3$. In the case when (4.123) holds, the inequality (4.122) holds for a constant $c = c(n, p, q, \nu, L)$ by (4.100) and (4.104). We now turn to the other case, that is, when (4.124) holds. Applying (4.100) and (4.106) with $K = 10^3$, we can assert that the inequality (4.122) holds for a constant $c = c(\mathtt{data})$. Therefore, in either case, (4.122) holds with a constant $c = c(\mathtt{data})$ as required.

Therefore, we conclude from (4.122) and Lemma 4.1.13 that

$$\sup_{B_i^{1+}} H(x_{i,M}, Dv_i) \leq c \fint_{B_i^{2+}} H(x_{i,M}, Dv_i)\, dx \leq c\lambda. \tag{4.125}$$

□

Remark 4.1.17. *For the interior case, Lemma 4.1.14 and Lemma 4.1.15 with B_i^{j+} replaced by B_i^{j-1} still hold. Furthermore, we have the interior Lipschitz regularity for the solution v_i to the reference problem (4.99). In fact, it follows from Lemma 4.1.12 and (4.122) that*

$$\sup_{B_i^1} H(x_{i,M}, Dv_i) \leq c\lambda, \tag{4.126}$$

where $c = c(\mathtt{data})$ is a positive constant.

4.1.4 Proof of Theorem 4.1.2

The comparison results which we have proved in the previous subsection and a more sophisticated covering argument enable us to prove the following proposition.

Proposition 4.1.18. *Let $u \in W^{1,1}(B_R^+)$ be a distributional solution to (4.67) with (4.68), under the assumptions (4.2) and (1.13). If $H(x, F) \in L^\gamma(B_R^+)$ for some $\gamma \in (1, \infty)$, then $H(x, Du) \in L^\gamma(B_{R/2}^+)$ with the estimate*

$$\fint_{B_{R/2}^+} [H(x, Du)]^\gamma\, dx \leq c \left(\fint_{B_R^+} H(x, Du)\, dx \right)^\gamma + c \fint_{B_R^+} [H(x, F)]^\gamma\, dx, \tag{4.127}$$

where $c = c(\texttt{data}, \gamma)$ *is a positive constant.*

Proof. We begin by recalling the covering argument presented in the preceding subsection. According to the covering argument, there exists a countable family of disjoint sets $\{B^+_{\rho_i}(y_i)\}_{i=1}^{\infty}$ satisfying (4.75) and (4.76). Let us first estimate $|B^+_{\rho_i}(y_i)|$. In light of (4.76), we have

$$\begin{aligned}
\lambda|B^+_{\rho_i}(y_i)| &= \int_{B^+_{\rho_i}(y_i)} \left[H(x,Du) + \frac{1}{\delta}H(x,F)\right] dx \\
&\leq \int_{\{x\in B^+_{\rho_i}(y_i):H(x,Du(x))>\frac{\lambda}{4}\}} H(x,Du(x))\,dx + \frac{\lambda}{4}|B^+_{\rho_i}(y_i)| \\
&\quad + \frac{1}{\delta}\int_{\{x\in B^+_{\rho_i}(y_i):H(x,F(x))>\frac{\delta\lambda}{4}\}} H(x,F(x))\,dx + \frac{\lambda}{4}|B^+_{\rho_i}(y_i)|,
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\lambda}{2}|B^+_{\rho_i}(y_i)| &\leq \int_{\{x\in B^+_{\rho_i}(y_i):H(x,Du)>\frac{\lambda}{4}\}} H(x,Du)\,dx \\
&\quad + \frac{1}{\delta}\int_{\{x\in B^+_{\rho_i}(y_i):H(x,F)>\frac{\delta\lambda}{4}\}} H(x,F)\,dx. \qquad (4.128)
\end{aligned}$$

We note that for any natural number N,

$$\begin{aligned}
H(x,\xi_1+\cdots+\xi_N) &= |\xi_1+\cdots+\xi_N|^p + a(x)|\xi_1+\cdots+\xi_N|^q \\
&\leq N^{p-1}(|\xi_1|^p+\cdots+|\xi_N|^p) + a(x)N^{q-1}(|\xi_1|^q+\cdots+|\xi_N|^q) \\
&\leq N^{q-1}\left[|\xi_1|^p+\cdots+|\xi_N|^p + a(x)(|\xi_1|^q+\cdots+|\xi_N|^q)\right] \\
&= N^{q-1}\left[H(x,\xi_1)+\cdots+H(x,\xi_N)\right] \qquad (4.129)
\end{aligned}$$

whenever $x \in \Omega$ and $\xi_1, \cdots, \xi_N \in \mathbb{R}^n$. As in the previous subsection, we consider the interior case and the boundary case separately. For the interior case, we use (4.97), (4.126) and (4.129), to find that

$$\begin{aligned}
&3^{q-1}c_1\lambda|\{x\in B_{5\rho_i}(y_i) : H(x,Du) > 2\cdot 3^{q-1}c_1\lambda\}| \\
&\quad + \frac{1}{2}\int_{\{x\in B_{5\rho_i}(y_i):H(x,Du)>2\cdot 3^{q-1}c_1\lambda\}} H(x,Du)\,dx \\
&\leq \int_{\{x\in B_{5\rho_i}(y_i):H(x,Du)>2\cdot 3^{q-1}c_1\lambda\}} H(x,Du)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le 3^{q-1}\Bigg[\int_{B_{5\rho_i}(y_i)} H(x, Du - Dw_i)\,dx + \int_{B_{5\rho_i}(y_i)} H(x_{i,M}, Dw_i - Dv_i)\,dx \\
&\qquad + \int_{\{x\in B_{5\rho_i}(y_i): H(x,Du)>2\cdot 3^{q-1}c_1\lambda\}} H(x_{i,M}, Dv_i)\,dx\Bigg] \\
&\le 3^{q-1}\Bigg[\int_{B_{5\rho_i}(y_i)} H(x, Du - Dw_i)\,dx + \int_{B_{5\rho_i}(y_i)} H(x_{i,M}, Dw_i - Dv_i)\,dx\Bigg] \\
&\quad + 3^{q-1}c_1\lambda|\{x\in B_{5\rho_i}(y_i) : H(x,Du) > 2\cdot 3^{q-1}c_1\lambda\}|, \qquad (4.130)
\end{aligned}
$$

where $c_1 \equiv c_1(\mathtt{data})$ is the constant appearing in (4.126). Then it follows from (4.130), (4.77), (4.78), (4.88), (4.103) and (4.105) with Remark 4.1.17 that

$$
\begin{aligned}
&\int_{\{x\in B_{5\rho_i}(y_i): H(x,Du)>2\cdot 3^{q-1}c_1\lambda\}} H(x, Du)\,dx \\
&\le 2\cdot 3^{q-1}\Bigg[\int_{B_{5\rho_i}(y_i)} H(x, Du - Dw_i)\,dx + \int_{B_{5\rho_i}(y_i)} H(x_{i,M}, Dw_i - Dv_i)\,dx\Bigg] \\
&\le 2\cdot 3^{q-1}\Bigg[\int_{B_i^3} H(x, Du - Dw_i)\,dx + \int_{B_i^2} H(x_{i,M}, Dw_i - Dv_i)\,dx\Bigg] \\
&\le 2\cdot 3^{q-1}|B_i^3|\Bigg[\fint_{B_i^3} H(x, Du - Dw_i)\,dx + \fint_{B_i^2} H(x_{i,M}, Dw_i - Dv_i)\,dx\Bigg] \\
&\le 2\cdot 3^{q-1}|B_i^3|\left[\varepsilon + \frac{\overline{c}}{K^{\frac{p-1}{2p-1}}} + c^*(K)\rho_i^{\sigma_0}\right]\lambda \\
&\le 2\cdot 3^{q-1}\cdot 15^n|B_i^0|\left[\varepsilon + \frac{\overline{c}}{K^{\frac{p-1}{2p-1}}} + c^*(K)\widetilde{R}^{\sigma_0}\right]\lambda, \qquad (4.131)
\end{aligned}
$$

where $\overline{c} \equiv \overline{c}(n, p, q, \nu, L)$ and $c^*(K) \equiv c^*(\mathtt{data}, K)$ are the constants appearing in (4.103) and (4.105) respectively.

Now for the boundary case, using (4.97), (4.121) and (4.129), we have

$$
\begin{aligned}
&3^{q-1}c_2\lambda|\{x\in B^+_{5\rho_i}(y_i) : H(x,Du) > 2\cdot 3^{q-1}c_2\lambda\}| \\
&\quad + \frac{1}{2}\int_{\{x\in B^+_{5\rho_i}(y_i): H(x,Du)>2\cdot 3^{q-1}c_2\lambda\}} H(x, Du)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{\{x\in B^+_{5\rho_i}(y_i):H(x,Du)>2\cdot 3^{q-1}c_1\lambda\}} H(x,Du)\,dx \\
&\le 3^{q-1}\Bigg[\int_{B^+_{5\rho_i}(y_i)} H(x,Du-Dw_i)\,dx + \int_{B^+_{5\rho_i}(y_i)} H(x_{i,M},Dw_i-Dv_i)\,dx \\
&\qquad + \int_{\{x\in B^+_{5\rho_i}(y_i):H(x,Du)>2\cdot 3^{q-1}c_1\lambda\}} H(x_{i,M},Dv_i)\,dx\Bigg] \\
&\le 3^{q-1}\left[\int_{B^+_{5\rho_i}(y_i)} H(x,Du-Dw_i)\,dx + \int_{B^+_{5\rho_i}(y_i)} H(x_{i,M},Dw_i-Dv_i)\,dx\right] \\
&\quad + 3^{q-1}c_2\lambda|\{x\in B^+_{5\rho_i}(y_i): H(x,Du)>2\cdot 3^{q-1}c_2\lambda\}|,
\end{aligned}
\tag{4.132}
$$

where $c_2 \equiv c_2(\texttt{data})$ is the constant appearing in (4.121). From (4.132), (4.78), (4.81) and the comparison estimates (4.88), (4.103) and (4.105), we obtain

$$
\begin{aligned}
&\int_{\{x\in B^+_{5\rho_i}(y_i):H(x,Du)>2\cdot 3^{q-1}c_2\lambda\}} H(x,Du)\,dx \\
&\le 2\cdot 3^{q-1}\left[\int_{B^+_{5\rho_i}(y_i)} H(x,Du-Dw_i)\,dx + \int_{B^+_{5\rho_i}(y_i)} H(x_{i,M},Dw_i-Dv_i)\,dx\right] \\
&\le 3\cdot 3^{q-1}|B_i^{4+}|\left[\fint_{B_i^{4+}} H(x,Du-Dw_i)\,dx + \fint_{B_i^{3+}} H(x_{i,M},Dw_i-Dv_i)\,dx\right] \\
&\le 2\cdot 3^{q-1}|B_i^{4+}|\left[\varepsilon + \frac{\overline{c}}{K^{\frac{p-1}{2p-1}}} + c^*(K)\rho_i^{\sigma_0}\right]\lambda \\
&\le 2\cdot 3^{q-1}\cdot 80^n|B_i^{0+}|\left[\varepsilon + \frac{\overline{c}}{K^{\frac{p-1}{2p-1}}} + c^*(K)\widetilde{R}^{\sigma_0}\right]\lambda.
\end{aligned}
\tag{4.133}
$$

It is clear that

$$
\int_{\{H(x,Du)>k_1\}} H(x,Du)\,dx \le \int_{\{H(x,Du)>k_2\}} H(x,Du)\,dx \quad \text{if} \quad k_1\ge k_2>0. \tag{4.134}
$$

Therefore, in either case, we deduce from (4.131), (4.133) and (4.134) that

$$\int_{\{x\in B^+_{5\rho_i}(y_i):H(x,Du)>c_3\lambda\}} H(x,Du)\,dx \le 2\cdot 3^{q-1}\cdot 80^n\, C(\varepsilon,K,\widetilde{R})\lambda|B^+_{\rho_i}(y_i)|, \tag{4.135}$$

where $c_3 = 2\cdot 3^{q-1}\max\{c_1,c_2\}$ is a positive constant depending on `data` and

$$C(\varepsilon,K,\widetilde{R}) \equiv \varepsilon + \frac{\bar{c}}{K^{\frac{p-1}{2p-1}}} + c^*(K)\widetilde{R}^{\sigma_0}. \tag{4.136}$$

Combining (4.128) and (4.135) yields

$$\begin{aligned}
&\int_{\{x\in B^+_{5\rho_i}(y_i):H(x,Du)>c_3\lambda\}} H(x,Du)\,dx \\
&\le 4^q\cdot 80^n\, C(\varepsilon,K,\widetilde{R}) \int_{\{x\in B^+_{\rho_i}(y_i):H(x,Du)>\frac{\lambda}{4}\}} H(x,Du)\,dx \\
&\quad + 4^q\cdot 80^n\, C(\varepsilon,K,\widetilde{R})\,\frac{1}{\delta} \int_{\{x\in B^+_{\rho_i}(y_i):H(x,F)>\frac{\delta\lambda}{4}\}} H(x,F)\,dx.
\end{aligned} \tag{4.137}$$

Hereafter for the sake of simplicity as in (4.69), we denote the upper level set of $H(x,F)$ by

$$\Xi(\lambda,s) := \left\{x\in B^+_s : H(x,F(x))>\lambda\right\}, \quad \frac{R}{2}\le s\le R, \quad \lambda>0. \tag{4.138}$$

Since $c_3>1$ and the family $\{B^+_{\rho_i}(y_i)\}_{i=1}^\infty$ is disjoint, it follows from (4.75), (4.78), (4.134) and (4.137) that

$$\begin{aligned}
\int_{E(c_3\lambda,r_1)} H(x,Du)\,dx &\le \sum_{i\ge1}\int_{\{x\in B^+_{5\rho_i}(y_i):H(x,Du)>c_3\lambda\}} H(x,Du)\,dx \\
&\le 4^q\cdot 80^n\, C(\varepsilon,K,\widetilde{R})\sum_{i\ge1}\int_{\{x\in B^+_{\rho_i}(y_i):H(x,Du)>\frac{\lambda}{4}\}} H(x,Du)\,dx \\
&\quad + 4^q\cdot 80^n\, C(\varepsilon,K,\widetilde{R})\,\frac{1}{\delta}\sum_{i\ge1}\int_{\{x\in B^+_{\rho_i}(y_i):H(x,F)>\frac{\delta\lambda}{4}\}} H(x,F)\,dx \\
&= 4^q\cdot 80^n\, C(\varepsilon,K,\widetilde{R})\int_{\bigcup_{i\ge1}\{x\in B^+_{\rho_i}(y_i):H(x,Du)>\frac{\lambda}{4}\}} H(x,Du)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ 4^q \cdot 80^n \, C(\varepsilon, K, \widetilde{R}) \, \frac{1}{\delta} \int_{\bigcup_{i\geq 1}\{x \in B^+_{\rho_i}(y_i) : H(x,F) > \frac{\delta\lambda}{4}\}} H(x,F)\, dx \\
\leq\ & 4^q \cdot 80^n \, C(\varepsilon, K, \widetilde{R}) \int_{E\left(\frac{\lambda}{4}, r_2\right)} H(x, Du)\, dx \\
&+ 4^q \cdot 80^n \, C(\varepsilon, K, \widetilde{R}) \, \frac{1}{\delta} \int_{\Xi\left(\frac{\delta\lambda}{4}, r_2\right)} H(x,F)\, dx.
\end{aligned}
$$

After a change of variable with respect to λ, we conclude that

$$
\begin{aligned}
&\int_{E(\lambda, r_1)} H(x, Du)\, dx \\
&\qquad \leq 4^q \cdot 80^n \, C(\varepsilon, K, \widetilde{R}) \int_{E\left(\frac{\lambda}{4c_3}, r_2\right)} H(x, Du)\, dx \\
&\qquad\quad + 4^q \cdot 80^n \, C(\varepsilon, K, \widetilde{R}) \, \frac{1}{\delta} \int_{\Xi\left(\frac{\delta\lambda}{4c_3}, r_2\right)} H(x,F)\, dx, \qquad (4.139)
\end{aligned}
$$

whenever $\lambda > c_3\lambda_0$.

Now, by Fubini's theorem, we see that

$$
(\gamma - 1) \int_0^M \lambda^{\gamma-2} \int_{E(\lambda, r_1)} H(x, Du)\, dx\, d\lambda = \int_{B^+_{r_1}} H(x, Du)[H(x, Du)_M]^{\gamma-1}\, dx \qquad (4.140)
$$

for any $M > 0$, where $H(x, Du)_M := \min\{H(x, Du), M\}$ is the truncated function of $H(x, Du)$. Here, note that the right-hand side of (4.140) is finite since $H(x, Du)_M$ is bounded, $\gamma > 1$ and $H(x, Du) \in L^1(\Omega)$. Hence for $M > c_3\lambda_0$, we have

$$
\begin{aligned}
\int_{B^+_{r_1}} H(x, Du)[H(x, Du)_M]^{\gamma-1}\, dx &= (\gamma - 1) \int_0^{c_3\lambda_0} \lambda^{\gamma-2} \int_{E(\lambda, r_1)} H(x, Du)\, dx\, d\lambda \\
&\quad + (\gamma - 1) \int_{c_3\lambda_0}^M \lambda^{\gamma-2} \int_{E(\lambda, r_1)} H(x, Du)\, dx\, d\lambda \\
&=: \mathrm{I}_1 + \mathrm{I}_2. \qquad (4.141)
\end{aligned}
$$

We estimate I_1 as follows:

$$\begin{aligned}
\mathrm{I}_1 &= (\gamma-1)\int_0^{c_3\lambda_0} \lambda^{\gamma-2} \int_{E(\lambda,r_1)} H(x,Du)\,dx\,d\lambda \\
&\le (\gamma-1)\int_0^{c_3\lambda_0} \lambda^{\gamma-2} d\lambda \int_{B_R^+} H(x,Du)\,dx \\
&= (c_3\lambda_0)^{\gamma-1} \int_{B_R^+} H(x,Du)\,dx. \qquad (4.142)
\end{aligned}$$

Using (4.139), we next estimate I_2 as follows:

$$\begin{aligned}
\mathrm{I}_2 &= (\gamma-1)\int_{c_3\lambda_0}^{M} \lambda^{\gamma-2} \int_{E(\lambda,r_1)} H(x,Du)\,dx\,d\lambda \\
&\le 4^q\cdot 80^n\, C(\varepsilon,K,\widetilde{R}) \Bigg((\gamma-1)\int_{c_3\lambda_0}^{M} \lambda^{\gamma-2} \int_{E\left(\frac{\lambda}{4c_3},r_2\right)} H(x,Du)\,dx\,d\lambda \\
&\qquad\qquad + \frac{1}{\delta}\cdot(\gamma-1)\int_{c_3\lambda_0}^{M} \lambda^{\gamma-2} \int_{\Xi\left(\frac{\delta\lambda}{4c_3},r_2\right)} H(x,F)\,dx\,d\lambda \Bigg) \\
&=: 4^q\cdot 80^n\, C(\varepsilon,K,\widetilde{R})\left(\mathrm{II}_1+\mathrm{II}_2\right). \qquad (4.143)
\end{aligned}$$

By a change of variable and Fubini's theorem, we estimate II_1 and II_2 as follows:

$$\begin{aligned}
\mathrm{II}_1 &= (\gamma-1)\int_{c_3\lambda_0}^{M} \lambda^{\gamma-2} \int_{E\left(\frac{\lambda}{4c_3},r_2\right)} H(x,Du)\,dx\,d\lambda \\
&\le (\gamma-1)\int_0^{M} \lambda^{\gamma-2} \int_{E\left(\frac{\lambda}{4c_3},r_2\right)} H(x,Du)\,dx\,d\lambda \\
&= (4c_3)^{\gamma-1}(\gamma-1)\int_0^{\frac{M}{4c_3}} \lambda^{\gamma-2} \int_{E(\lambda,r_2)} H(x,Du)\,dx\,d\lambda \\
&\le (4c_3)^{\gamma-1}(\gamma-1)\int_0^{M} \lambda^{\gamma-2} \int_{E(\lambda,r_2)} H(x,Du)\,dx\,d\lambda \\
&= (4c_3)^{\gamma-1} \int_{B_{r_2}^+} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx, \qquad (4.144)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{II}_2 &= \frac{1}{\delta} \cdot (\gamma - 1) \int_{c_3\lambda_0}^{M} \lambda^{\gamma-2} \int_{\Xi\left(\frac{\delta\lambda}{4c_3}, r_2\right)} H(x,F)\, dx\, d\lambda \\
&\leq \frac{1}{\delta} \cdot (\gamma - 1) \int_{0}^{\infty} \lambda^{\gamma-2} \int_{\Xi\left(\frac{\delta\lambda}{4c_3}, r_2\right)} H(x,F)\, dx\, d\lambda \\
&= \frac{1}{\delta} \left(\frac{4c_3}{\delta}\right)^{\gamma-1} (\gamma - 1) \int_{0}^{\infty} \lambda^{\gamma-2} \int_{\Xi(\lambda, r_2)} H(x,F)\, dx\, d\lambda \\
&= \frac{1}{\delta} \left(\frac{4c_3}{\delta}\right)^{\gamma-1} \int_{B_{r_2}^+} [H(x,F)]^{\gamma}\, dx. \qquad (4.145)
\end{aligned}$$

Combining (4.141)-(4.145) gives

$$\begin{aligned}
&\int_{B_{r_1}^+} H(x,Du)[H(x,Du)_M]^{\gamma-1}\, dx \\
&\quad \leq (c_3\lambda_0)^{\gamma-1} |B_R^+| \fint_{B_R^+} H(x,Du)\, dx + 4^q \cdot 80^n \cdot (4c_3)^{\gamma-1}\, C(\varepsilon, K, \widetilde{R}) \\
&\qquad \times \left(\int_{B_{r_2}^+} H(x,Du)[H(x,Du)_M]^{\gamma-1}\, dx + \frac{1}{\delta^{\gamma}} \int_{B_{r_2}^+} [H(x,F)]^{\gamma}\, dx \right) \\
&\quad = (c_3\lambda_0)^{\gamma-1} |B_R^+| \fint_{B_R^+} H(x,Du)\, dx \\
&\qquad + 4^q \cdot 80^n \cdot (4c_3)^{\gamma-1} \left[\varepsilon + \frac{\overline{c}}{K^{\frac{p-1}{2p-1}}} + c^*(K)\widetilde{R}^{\sigma_0} \right] \\
&\qquad \times \left(\int_{B_{r_2}^+} H(x,Du)[H(x,Du)_M]^{\gamma-1}\, dx + \frac{1}{\delta^{\gamma}} \int_{B_{r_2}^+} [H(x,F)]^{\gamma}\, dx \right).
\end{aligned} \qquad (4.146)$$

We now take ε, K and $\widetilde{R}$ in order to obtain

$$4^q \cdot 80^n \cdot (4c_3)^{\gamma-1} \left[\varepsilon + \frac{\overline{c}}{K^{\frac{p-1}{2p-1}}} + c^*(K)\widetilde{R}^{\sigma_0} \right] \leq \frac{1}{2}. \qquad (4.147)$$

To put it concretely, we first choose $\varepsilon \equiv \varepsilon(\mathtt{data}, \gamma) \in (0,1)$ and $K \equiv$

$K(\mathtt{data},\gamma) > 240$ in order that

$$\varepsilon \le \frac{1}{4^q \cdot 80^n \cdot (4c_3)^{\gamma-1}} \cdot \frac{1}{6} \quad \text{and} \quad K \ge \left(6 \cdot 4^q \cdot 80^n \cdot (4c_3)^{\gamma-1}\bar{c}\right)^{\frac{2p-1}{p-1}}.$$

Then $c^* \equiv c^*(K) \equiv c^*(\mathtt{data}, K)$ is the constant depending on $\mathtt{data}$ and γ. Hence we next take $\widetilde{R} \equiv \widetilde{R}(\mathtt{data},\gamma) \in (0,1)$ in order that

$$\widetilde{R} \le \left(\frac{1}{4^q \cdot 90^n \cdot (4c_3)^{\gamma-1}c^*} \cdot \frac{1}{6}\right)^{\frac{1}{\sigma_0}}.$$

Note that once $\varepsilon \equiv \varepsilon(\mathtt{data},\gamma) \in (0,1)$ is chosen, one can find a corresponding constant $\delta \equiv \delta(\mathtt{data},\gamma) > 0$. Recalling the definition of λ_0 in (4.72), we deduce that

$$\begin{aligned}
\lambda_0 &= \frac{100^n r_2^n}{(r_2-r_1)^n} \fint_{B_{r_2}^+} \left[H(x,Du) + \frac{1}{\delta} H(x,F)\right] dx \\
&= \frac{100^n r_2^n}{(r_2-r_1)^n} \frac{|B_R^+|}{|B_{r_2}^+|} \fint_{B_R^+} \left[H(x,Du) + \frac{1}{\delta} H(x,F)\right] dx \\
&= \frac{100^n R^n}{(r_2-r_1)^n} \fint_{B_R^+} \left[H(x,Du) + \frac{1}{\delta} H(x,F)\right] dx. \qquad (4.148)
\end{aligned}$$

Consequently, we discover from (4.146)-(4.148) that

$$\begin{aligned}
&\int_{B_{r_1}^+} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\
&\qquad \le \frac{1}{2}\int_{B_{r_2}^+} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\
&\qquad\quad + \frac{c_4^{\gamma-1} R^{n(\gamma-1)}|B_R^+|}{(r_2-r_1)^{n(\gamma-1)}} \left(\fint_{B_R^+} \left[H(x,Du) + \frac{1}{\delta} H(x,F)\right] dx\right)^{\gamma} \\
&\qquad\quad + \frac{1}{\delta^\gamma}\int_{B_R^+} [H(x,F)]^\gamma\,dx, \qquad (4.149)
\end{aligned}$$

where $c_4 \equiv c_4(\mathtt{data})$ and $\delta \equiv \delta(\mathtt{data},\gamma)$ are positive constants.

We now apply Lemma 2.3.1 with

$$\phi(s) = \int_{B_s^+} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx, \quad \kappa = n(\gamma-1) > 0 \quad \text{and} \quad \vartheta = \frac{1}{2}$$

to derive that for some constant $c \equiv c(\mathtt{data},\gamma)$,

$$\begin{aligned}\int_{B_{R/2}^+} & H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\ & \le c|B_R^+|\left(\fint_{B_R^+}[H(x,Du)+H(x,F)]\,dx\right)^{\gamma} + c\int_{B_R^+}[H(x,F)]^{\gamma}\,dx\end{aligned}$$

and so

$$\begin{aligned}\fint_{B_{R/2}^+} & H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\ & \le c\left(\fint_{B_R^+} H(x,Du)\,dx\right)^{\gamma} + c\fint_{B_R^+}[H(x,F)]^{\gamma}\,dx.\end{aligned}$$

By Fatou's lemma, we conclude that

$$\fint_{B_{R/2}^+}[H(x,Du)]^{\gamma}\,dx \le c\left(\fint_{B_R^+} H(x,Du)\,dx\right)^{\gamma} + c\fint_{B_R^+}[H(x,F)]^{\gamma}\,dx.$$

□

The following interior estimate can be derived in the same way to the boundary estimate (4.127), see also [40, Theorem 1.1].

Proposition 4.1.19. *Let $u \in W^{1,1}(B_R)$ be a distributional solution to*

$$\operatorname{div} A(x,Du) = \operatorname{div} G(x,F) \quad \text{in } B_R,$$

with $H(x,Du),\ H(x,F) \in L^1(B_R)$, under the assumptions (4.2) and (1.13). If $H(x,F) \in L^{\gamma}(B_R)$ for some $\gamma \in (1,\infty)$, then $H(x,Du) \in L^{\gamma}(B_{R/2})$ with *the estimate*

$$\fint_{B_{R/2}} [H(x, Du)]^{\gamma}\, dx \le c \left(\fint_{B_R} H(x, Du)\, dx \right)^{\gamma} + c \fint_{B_R} [H(x, F)]^{\gamma}\, dx, \tag{4.150}$$

where $c = c(\mathtt{data}, \gamma)$ *is a positive constant.*

We now prove the main result via a standard covering and flattening argument.

Proof of Theorem 4.1.2. Since Ω is bounded in $\mathbb{R}^n$, there exists a covering of Ω with a finite number of balls such that the balls touching the boundary $\partial\Omega$ are centered on the boundary itself and the radius R of the interior balls satisfies $R \le \widetilde{R}$, where $\widetilde{R} \equiv \widetilde{R}(\mathtt{data}, \gamma)$ is the small number found in the proof of Proposition 4.1.18. Since $\partial\Omega$ is compact and of class $C^{1,\beta}$, there exists a family of $C^{1,\beta}$-regular maps $\{\Psi_j\}_{j=1}^N$, where $\Psi_j : B^+_{\widetilde{R}} \to \Omega$ with $\Psi_j(T_{\widetilde{R}}) \subset \partial\Omega$, $\det D\Psi_j \equiv 1$ and $\partial\Omega \subset \bigcup_{j=1}^N \Psi_j(T_{\widetilde{R}})$. For simplicity of notation, we write Ψ instead of Ψ_j. Then we define

$$\begin{aligned}
&\widetilde{u}(x) := u(\Psi(x)), \qquad \widetilde{a}(x) := a(\Psi(x)), \qquad \widetilde{F}(x) := F(\Psi(x)),\\
&\widetilde{A}(x, \xi) := A(\Psi(x), \xi D(\Psi^{-1})(x))\, [D(\Psi^{-1})(x)]^T,\\
&\widetilde{G}(x, \xi) := G(\Psi(x), \xi)\, [D(\Psi^{-1})(x)]^T.
\end{aligned}$$

Since Ψ is $C^{1,\beta}$-regular with $\beta \ge \alpha$, the new vector fields $\widetilde{A}$ and $\widetilde{G}$ satisfy the following structural conditions:

$$|\widetilde{A}(x, \xi)| + |\xi||D_\xi \widetilde{A}(x, \xi)| \le \widetilde{L} \left(|\xi|^{p-1} + \widetilde{a}(x)|\xi|^{q-1} \right), \tag{4.151}$$

$$\widetilde{\nu} \left(|\xi|^{p-2} + \widetilde{a}(x)|\xi|^{q-2} \right) |\eta|^2 \le \langle D_\xi \widetilde{A}(x, \xi)\eta, \eta \rangle, \tag{4.152}$$

$$|\widetilde{A}(x_1, \xi) - \widetilde{A}(x_2, \xi)| \le \widetilde{L}\, \omega_\alpha(|x_1 - x_2|) \left(|\xi|^{p-1} + |\xi|^{q-1} \right), \tag{4.153}$$

$$|\widetilde{G}(x, \xi)| \le \widetilde{L} \left(|\xi|^{p-1} + \widetilde{a}(x)|\xi|^{q-1} \right), \tag{4.154}$$

with

$$\omega_\alpha(s) := \min\{s^\alpha, 1\},$$

where $\widetilde{\nu}$ and $\widetilde{L}$ are positive constants depending on $n, \nu, L, \|a\|_{L^\infty(\Omega)}, [a]_{C^{0,\alpha}(\Omega)}$ and the chart Ψ, see [81] for more details. We notice that ellipticity and growth conditions (4.151), (4.152) and (4.154) are the same as (4.5), (4.6) and (4.8), respectively, and that the condition (4.153) is slightly different from (4.7). Nevertheless, Lemmas 4.1.9, 4.1.10 and 4.1.15 continue to hold under the structural conditions (4.151)-(4.154), with minor modifications in the proofs. Applying Proposition 4.1.18, and then going back to the original system, we obtain that

$$\begin{aligned}\int_{V_j}[H(x,Du)]^\gamma\,dx &\le c\widetilde{R}^{n(1-\gamma)}\left(\int_{U_j}H(x,Du)\,dx\right)^\gamma + c\int_{U_j}[H(x,F)]^\gamma\,dx\\ &\le c\widetilde{R}^{n(1-\gamma)}\left(\int_{\Omega}H(x,Du)\,dx\right)^\gamma + c\int_{\Omega}[H(x,F)]^\gamma\,dx,\end{aligned} \tag{4.155}$$

for $j=1,\cdots,N$, where $V_j := \Psi_j(B^+_{\widetilde{R}/2})$ and $U_j := \Psi_j(B^+_{\widetilde{R}})$. Note that the constant $c \equiv c(\texttt{data},\gamma)$ in the above display is independent of j. Similarly, we deduce from Proposition 4.1.19 and a standard covering argument that

$$\int_{V}[H(x,Du)]^\gamma\,dx \le c\widetilde{R}^{n(1-\gamma)}\left(\int_{\Omega}H(x,Du)\,dx\right)^\gamma + c\int_{\Omega}[H(x,F)]^\gamma\,dx, \tag{4.156}$$

for some positive constant $c = c(\texttt{data},\gamma,\Omega)$, where $V \Subset \Omega$ is an open set with $\Omega \subset V \cup \left(\bigcup_{j=1}^N V_j\right)$. We combine (4.155) and (4.156) to discover that

$$\begin{aligned}\int_{\Omega}[H(x,Du)]^\gamma\,dx &\le \int_{V}[H(x,Du)]^\gamma\,dx + \sum_{j=1}^N\int_{V_j}[H(x,Du)]^\gamma\,dx\\ &\le c\widetilde{R}^{n(1-\gamma)}\left(\int_{\Omega}H(x,Du)\,dx\right)^\gamma + c\int_{\Omega}[H(x,F)]^\gamma\,dx\\ &\le c\widetilde{R}^{n(1-\gamma)}\left(\int_{\Omega}H(x,F)\,dx\right)^\gamma + c\int_{\Omega}[H(x,F)]^\gamma\,dx\\ &\le c\left(\widetilde{R}^{n(1-\gamma)}|\Omega|^{\gamma-1}+1\right)\int_{\Omega}[H(x,F)]^\gamma\,dx,\end{aligned}$$

where we have used the energy estimate (4.20) and Hölder's inequality. The desired estimate (4.16) now follows. □

4.2 Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains

This section is concerned with the borderline case of double phase problems in divergence form. The problem under consideration is given by

$$\begin{aligned}\operatorname{div}\big(\beta(x)\left[|Du|^{p-2}Du + a(x)|Du|^{p-2}\log(e+|Du|)Du\right]\big)\\ = \operatorname{div}\big(|F|^{p-2}F + a(x)|F|^{p-2}\log(e+|F|)F\big) \quad \text{in } \Omega, \quad (4.157)\end{aligned}$$

where $1 < p < \infty$ is a fixed number, $\Omega \subset \mathbb{R}^n$ is a bounded open domain with $n \geq 2$, and $F : \Omega \to \mathbb{R}^n$ is a given vector field. Here the coefficient function $\beta : \mathbb{R}^n \to \mathbb{R}$ satisfies $\nu \leq \beta(\cdot) \leq L$ for some fixed constants $0 < \nu \leq L < \infty$, and the function $a : \mathbb{R}^n \to \mathbb{R}$ is always assumed to be non-negative and bounded.

The main object of this section is to investigate optimal conditions on the coefficient functions $\beta(\cdot), a(\cdot)$ and a minimal geometric assumption on $\partial\Omega$ under which the natural Calderón-Zygmund type relation

$$\begin{aligned}|F|^p + a(x)|F|^p\log(e+|F|) \in L^\gamma(\Omega)\\ \Longrightarrow\ |Du|^p + a(x)|Du|^p\log(e+|Du|) \in L^\gamma(\Omega) \quad (4.158)\end{aligned}$$

and the corresponding global gradient estimate

$$\begin{aligned}\left(\int_\Omega \left[|Du|^p + a(x)|Du|^p\log(e+|Du|)\right]^\gamma dx\right)^{\frac{1}{\gamma}}\\ \leq c\left(\int_\Omega \left[|F|^p + a(x)|F|^p\log(e+|F|)\right]^\gamma dx\right)^{\frac{1}{\gamma}} \quad (4.159)\end{aligned}$$

hold true for every $\gamma \in [1, \infty)$.

4.2.1 Hypotheses and main results

We deal with a distributional solution of the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x, Du) &= \operatorname{div} G(x, F) \quad \text{in} \quad \Omega, \\ u &= 0 \qquad\qquad\quad\ \text{on} \quad \partial\Omega, \end{cases} \tag{4.160}$$

where $\partial\Omega$ is the boundary of Ω.

The above problem is a generic one whose model is given by (4.157). Throughout this section, the nonlinearity $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be measurable with respect to x, differentiable with respect to $\xi \neq 0$ and to satisfy the following structural conditions:

$$|A(x,\xi)| + |\xi||D_\xi A(x,\xi)| \leq L\left[|\xi|^{p-1} + a(x)|\xi|^{p-1}\log(e+|\xi|)\right], \tag{4.161}$$

$$\langle D_\xi A(x,\xi)\eta, \eta\rangle \geq \nu\left[|\xi|^{p-2} + a(x)|\xi|^{p-2}\log(e+|\xi|)\right]|\eta|^2, \tag{4.162}$$

for almost every $x \in \mathbb{R}$ and for all $\xi, \eta \in \mathbb{R}^n$, where $0 < \nu \leq L < +\infty$ are fixed constants and $a : \mathbb{R}^n \to \mathbb{R}$ is a non-negative and bounded function.

We define the auxiliary vector fields $V_p, V_{\log} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$V_p(\xi) := |\xi|^{\frac{p-2}{2}}\xi \quad \text{and} \quad V_{\log}(\xi) := \left(|\xi|^{p-2}\log(e+|\xi|) + \frac{|\xi|^{p-1}}{p(e+|\xi|)}\right)^{\frac{1}{2}}\xi,$$

whenever $\xi \in \mathbb{R}^n$. Then the structural condition (4.162) implies the following monotonicity property:

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle \geq \widetilde{\nu}\left[|V_p(\xi) - V_p(\eta)|^2 + a(x)|V_{\log}(\xi) - V_{\log}(\eta)|^2\right], \tag{4.163}$$

where $\widetilde{\nu}$ is a positive constant depending only on n, p and ν. In particular, for the case $p \geq 2$, the above monotonicity property can be reduced to

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle \geq \widetilde{\nu}\left[|\xi-\eta|^p + a(x)|\xi-\eta|^p\log(e+|\xi-\eta|)\right]. \tag{4.164}$$

In addition, we remark that $|V_p(\xi)|^2 = |\xi|^p$ and that $|V_{\log}(\xi)|^2$ is comparable to $|\xi|^p\log(e+|\xi|)$, more precisely,

$$|\xi|^p\log(e+|\xi|) \leq |V_{\log}(\xi)|^2 \leq 2|\xi|^p\log(e+|\xi|), \quad \forall \xi \in \mathbb{R}^n. \tag{4.165}$$

On the other hand, a vector field $G : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ in the nonhomogeneous term of (4.160) is assumed to be a Carathéodory function, namely, measurable in x and continuous in ξ, satisfying the following growth condition:

$$|G(x,\xi)| \leq L\left[|\xi|^{p-1} + a(x)|\xi|^{p-1}\log(e+|\xi|)\right]. \tag{4.166}$$

We now introduce a regularity assumption on the modulating coefficient function $a(\cdot)$. We note that the energy functional $\mathcal{H}$ in (1.18) is a borderline case of (p,q)-energy functionals $\mathcal{H}_{p,q}$ in (1.19). On the double phase problems with (p,q)-growth, it is necessary to assume that $a(\cdot)$ is Hölder continuous, see for instance [55, 61]. However, in the borderline case, this assumption should be superfluous, as the phase transition between $|Du|^p$ and $|Du|^p\log(e+|Du|)$ is less dramatic. Here, we assume that $a(\cdot)$ is only log-Hölder continuous. Specifically, $a(\cdot)$ is assumed to be a continuous map satisfying

$$|a(x_1) - a(x_2)| \leq \omega(|x_1 - x_2|) \tag{4.167}$$

whenever $x_1, x_2 \in \mathbb{R}$, where $\omega : [0,\infty) \to [0,1]$ is a non-decreasing and concave modulus of continuity with a decay of logarithmic type as follows:

$$\lim_{r\to 0} \omega(r)\log\left(\frac{1}{r}\right) = 0. \tag{4.168}$$

We next discuss a regularity assumption on the nonlinearity with respect to x-variable. We define

$$\begin{aligned}\Theta(A, B_r(y))(x) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \Bigg| &\frac{A(x,\xi)}{|\xi|^{p-1} + a(x)|\xi|^{p-1}\log(e+|\xi|)} \\ &- \left(\frac{A(\cdot,\xi)}{|\xi|^{p-1} + a(\cdot)|\xi|^{p-1}\log(e+|\xi|)}\right)_{B_r(y)}\Bigg|.\end{aligned} \tag{4.169}$$

Throughout this section, $0 < \delta < \frac{1}{8}$ is a small constant to be determined later. On the other hand, R_0 can be any positive number. Our main assumption on the nonlinearity is the following.

Definition 4.2.1. *We say that $A(x,\xi)$ is (δ, R_0)-vanishing if*

$$\sup_{0<r\leq R_0}\sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \Theta(A, B_r(y))(x)\,dx \leq \delta.$$

We remark that it follows from the growth condition (4.161) that for any $\kappa > 1$,

$$\sup_{0<r\le R_0} \sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \Theta^\kappa \, dx \le (2L)^{\kappa-1}\delta. \tag{4.170}$$

We now introduce a geometric assumption on the boundary of the domain.

Definition 4.2.2. *We say that Ω is (δ, R_0)-Reifenberg flat if for each $x \in \partial\Omega$ and for each $r \in (0, R_0]$, there exists a coordinate system $\{y^1, \cdots, y^n\}$ such that $x = 0$ in this coordinate system and that*

$$B_r(0) \cap \{y^n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y^n > -\delta r\}.$$

We remark that a Reifenberg flat domain can go beyond Lipschitz category, not necessarily given by graphs. Nevertheless, it satisfies the following measure density conditions:

$$\sup_{0<r\le R_0} \sup_{x\in\Omega} \frac{|B_r(x)|}{|B_r(x)\cap\Omega|} \le \left(\frac{2}{1-\delta}\right)^n \le \left(\frac{16}{7}\right)^n, \tag{4.171}$$

$$\inf_{0<r\le R_0} \inf_{x\in\Omega} \frac{|B_r(x)\cap\Omega^{\mathsf{c}}|}{|B_r(x)|} \ge \left(\frac{1-\delta}{2}\right)^n \ge \left(\frac{7}{16}\right)^n. \tag{4.172}$$

We refer to [29, 70, 85, 86, 93, 96, 102, 108, 118] and references therein regarding its properties and applications in analysis and geometry.

We next return to a distributional solution to (4.160) under consideration.

Definition 4.2.3. *We say that $u \in W_0^{1,1}(\Omega)$ is a distributional solution to (4.160) if it satisfies*

$$\int_\Omega \langle A(x, Du), D\varphi\rangle \, dx = \int_\Omega \langle G(x, F), D\varphi\rangle \, dx, \tag{4.173}$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

We clearly point out that if $u \in W_0^{1,1}(\Omega)$ is a distributional solution to (4.160), with the natural integrability assumption $H(x, Du)$, $H(x, F) \in L^1(\Omega)$, then (4.173) still holds for every function $\varphi \in W_0^{1,1}(\Omega)$ with $H(x, D\varphi) \in L^1(\Omega)$, see Corollary 4.2.11 in the next subsection. We will also present necessary issues of solutions including existence, uniqueness and standard energy estimate in the next subsection.

In the rest of the section we shall use the notation

$$H(x,\xi) := |\xi|^p + a(x)|\xi|^p \log(e+|\xi|), \tag{4.174}$$

for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. According to the above remark (4.165), we have

$$\frac{1}{2}\left[|V_p(\xi)|^2 + a(x)|V_{\log}(\xi)|^2\right] \le H(x,\xi) \le |V_p(\xi)|^2 + a(x)|V_{\log}(\xi)|^2. \tag{4.175}$$

We now state the main result in this section.

Theorem 4.2.4. *Let $u \in W_0^{1,1}(\Omega)$ be the distributional solution to (4.160), with*

$$H(x,Du),\ H(x,F) \in L^1(\Omega). \tag{4.176}$$

Suppose that

$$H(x,F) \in L^\gamma(\Omega) \quad \textit{for some } \gamma \in (1,\infty). \tag{4.177}$$

Then there exists a constant $\delta = \delta(n,p,\nu,L,\|a\|_{L^\infty(\Omega)},\|H(\cdot,F)\|_{L^1(\Omega)},\gamma) > 0$ such that if A is (δ,R_0)-vanishing and Ω is (δ,R_0)-Reifenberg flat for some $R_0 > 0$, then we have

$$H(x,Du) \in L^\gamma(\Omega)$$

with the estimate

$$\left(\int_\Omega [H(x,Du)]^\gamma \, dx\right)^{\frac{1}{\gamma}} \le c\left(\int_\Omega [H(x,F)]^\gamma \, dx\right)^{\frac{1}{\gamma}}, \tag{4.178}$$

where $c = c(n,p,\nu,L,\omega(\cdot),\|a\|_{L^\infty(\Omega)},\|H(\cdot,F)\|_{L^1(\Omega)},\gamma,R_0,\Omega)$ is a positive constant.

Remark 4.2.5. *Some research activities have been carried out on double phase problems in the borderline case, see for instance [7, 8]. However, there are few results regarding Calderón-Zygmund type estimate for those problems, as far as we are concerned. The above Calderón-Zygmund estimate (4.178) is global, and we also obtain a local estimate in the proof of Theorem 4.2.4, see (4.294) in Subsection 4.2.5.*

Remark 4.2.6. *It is worth pointing out that there is a close relationship between the double phase problem in the borderline case and the $p(x)$-Laplacian* *type problem whose energy functional is given by*

$$v \in W^{1,1}(\Omega) \ \mapsto \ \int_\Omega |Dv|^{p(x)}\,dx, \quad 1 < p(x) < \infty. \tag{4.179}$$

As shown in [3, 7, 8, 24], the regularity assumption on the modulating coefficient $a(x)$ is exactly the same as that on the variable exponent $p(x)$. In fact, the variable exponent $p(x)$ in the functional (4.179) yields a logarithmic perturbation in the gradient when the point x varies, which is linked to a double phase functional in the borderline case (1.18).

4.2.2 Auxiliary lemmas

From now on, for the sake of convenience, we employ the letter c to denote any universal constants which can be explicitly computed in terms of known quantities such as $n, p, \nu, L, \gamma, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}$ and $\omega(\cdot)$. Thus the exact value denoted by c might be different from line to line.

We first introduce the zero extension lemma and the McShane extension lemma.

Lemma 4.2.7. *[5] Let U be a bounded domain in $\mathbb{R}^n$, and let $v \in W_0^{1,p}(U)$ for some $p \geq 1$. Let $\widetilde{v}$ denote the zero extension of v and $\widetilde{Dv}$ denote the zero extension of Dv, that is,*

$$\widetilde{v}(x) := \begin{cases} v(x) & \text{if } x \in U, \\ 0 & \text{if } x \in U^{\mathsf{c}}, \end{cases} \qquad \widetilde{Dv}(x) := \begin{cases} Dv(x) & \text{if } x \in U, \\ 0 & \text{if } x \in U^{\mathsf{c}}. \end{cases}$$

Then $D\widetilde{v} = \widetilde{Dv}$ in the weak sense, and hence $\widetilde{v} \in W^{1,p}(\mathbb{R}^n)$.

Lemma 4.2.8. *[92] Assume $U \subset \mathbb{R}^n$, and let $b : U \to \mathbb{R}$ be a continuous function which has a modulus of continuity ω satisfying (4.168). Then there exists an extension $\widetilde{b} : \mathbb{R}^n \to \mathbb{R}$ of b such that $\widetilde{b}$ admits ω as a modulus of continuity and $\|\widetilde{b}\|_{L^\infty(\mathbb{R}^n)} = \|b\|_{L^\infty(U)}$.*

Remark 4.2.9. *Lemma 4.2.8 shows that it does not matter whether the coefficient function $a(\cdot)$ is defined in $\Omega \subset \mathbb{R}^n$ or in the whole domain $\mathbb{R}^n$. Hence we can assume that the coefficient function $a(\cdot)$ is defined in $\mathbb{R}^n$ as in the previous subsection. In addition, it follows from Lemma 4.2.7 that if $H(x, Du) \in L^1(\Omega)$ for $u \in W_0^{1,1}(\Omega)$, then we have $H(x, D\widetilde{u}) \in L^1(\mathbb{R}^n)$, where $\widetilde{u}$ is the zero extension of u.*

We next discuss distributional solutions. The following proposition provides a criterion for admissible test functions.

Proposition 4.2.10. *Let $B \subset \Omega$ be a ball and let $W : B \to \mathbb{R}^n$ be a measurable vector field such that $H(x, W) \in L^1(B)$ and which is a distributional solution to the equation*

$$\operatorname{div} S(x, W) = 0 \quad \text{in} \quad B. \tag{4.180}$$

Here we assume that the vector field $S : B \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the growth condition

$$|S(x, \xi)| \leq L \left[|\xi|^{p-1} + a(x)|\xi|^{p-1} \log(e + |\xi|) \right] \tag{4.181}$$

for almost every $x \in B$ and for all $\xi \in \mathbb{R}^n$, where $a(\cdot)$ has a modulus of continuity with a decay of logarithmic type (4.168). Then every function $\varphi \in W_0^{1,1}(B)$ with $H(x, D\varphi) \in L^1(B)$ satisfies

$$\int_B \langle S(x, W), D\varphi \rangle \, dx = 0. \tag{4.182}$$

Proof. We only need to consider the case $B = B_1(0) = B_1$ by dilation and translation. Let us begin by constructing a sequence of functions $\{\varphi_k\} \subset C_0^\infty(B_1)$ such that

$$D\varphi_k \longrightarrow D\varphi \ \text{ a.e.} \quad \text{and} \quad H(x, D\varphi_k) \longrightarrow H(x, D\varphi) \ \text{ in } \ L^1(B_1). \tag{4.183}$$

The following construction is adapted from [40, 55, 129]. By Lemma 4.2.8, the continuous function $a(\cdot)$ can be extended on $\mathbb{R}^n$, with the same modulus of continuity. For simplicity of notation, we continue to write $a(\cdot)$ for the extension. Also we can take the zero extension of φ by Lemma 4.2.7, and hence we can assume that $\varphi \in W^{1,1}(\mathbb{R}^n)$ with $H(x, D\varphi) \in L^1(\mathbb{R}^n)$. Let $\psi \in C_0^\infty(B_1)$ be a mollifier with $\psi \geq 0$, $\int_{\mathbb{R}^n} \psi \, dx = 1$, and set

$$\psi_r(x) := \frac{1}{r^n} \psi \left(\frac{x}{r} \right)$$

for $x \in B_r$ with $r > 0$. Then it is clear that $\psi_r \in C_0^\infty(B_r)$, $\int_{\mathbb{R}^n} \psi_r \, dx = 1$ and $0 \leq \psi_r \leq c(n) r^{-n}$. Now we define, for $0 < r < \frac{1}{4}$,

$$\widehat{a}_r(x) := a\left(\frac{x}{1 - 2r} \right), \quad \widehat{\varphi}_r(x) := \varphi\left(\frac{x}{1 - 2r} \right), \quad \varphi_r := \widehat{\varphi}_r * \psi_r \in C_0^\infty(B_{1-r}).$$

We also define

$$a_r(x) := \inf_{y \in B_r(x)} \widehat{a}_r(y), \quad H_r(x,\xi) := |\xi|^p + a_r(x)|\xi|^p \log(e+|\xi|)$$

for $x \in B_1$ and $\xi \in \mathbb{R}^n$. By Hölder's inequality, we have

$$|D\varphi_r(x)| = |D\widehat{\varphi}_r * \psi_r(x)| \le c \left(\int_{B_1} |D\widehat{\varphi}_r|^p \, dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \psi_r^{\frac{p}{p-1}} \, dy \right)^{\frac{p-1}{p}} \le c r^{-\frac{n}{p}}$$

for every $x \in B_1$. Then it follows from the definitions of $a_r(\cdot)$ and $\widehat{a}_r(\cdot)$ that

$$\begin{aligned} H(x, D\varphi_r(x)) &\le |a(x) - a_r(x)||D\varphi_r(x)|^p \log(e + |D\varphi_r(x)|) + H_r(x, D\varphi_r(x)) \\ &\le c\omega(r) \log(e + cr^{-\frac{n}{p}})|D\varphi_r(x)|^p + H_r(x, D\varphi_r(x)). \end{aligned}$$

Here we note that

$$\begin{aligned} \log(e + cr^{-\frac{n}{p}}) &\le c \log(e + r^{-\frac{n}{p}}) \\ &\le c \log(2r^{-\frac{n}{p}}) = c \left[\log 2 + \frac{n}{p} \log\left(\frac{1}{r}\right) \right] \\ &\le c \left[1 + \log\left(\frac{1}{r}\right) \right] \le 2c \log\left(\frac{1}{r}\right), \end{aligned}$$

if $r > 0$ is sufficiently small. Therefore, we see that

$$\begin{aligned} H(x, D\varphi_r(x)) &\le c\omega(r) \log\left(\frac{1}{r}\right) |D\varphi_r(x)|^p + H_r(x, D\varphi_r(x)) \\ &\le c H_r(x, D\varphi_r(x)), \end{aligned} \tag{4.184}$$

since $a(\cdot)$ has a modulus of continuity with a decay of logarithmic type (4.168). By Jensen's inequality, we now estimate the right-hand side of (4.184) as follows:

$$\begin{aligned} H_r(x, D\varphi_r(x)) &\le \int_{B_r(x)} H_r(x, D\widehat{\varphi}_r(y)) \psi_r(x-y) \, dy \\ &= \int_{B_r(x)} H_r\left(x, \frac{1}{1-2r} D\varphi\left(\frac{y}{1-2r}\right)\right) \psi_r(x-y) \, dy \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{(1-2r)^{p+1}} \int_{B_r(x)} H_r\left(x, D\varphi\left(\frac{y}{1-2r}\right)\right) \psi_r(x-y)\, dy \\
&\leq 2^{p+1} \int_{B_r(x)} H\left(\frac{y}{1-2r}, D\varphi\left(\frac{y}{1-2r}\right)\right) \psi_r(x-y)\, dy \\
&= 2^{p+1} \left[H\left(\frac{\cdot}{1-2r}, D\varphi\left(\frac{\cdot}{1-2r}\right)\right) * \psi_r\right](x). \qquad (4.185)
\end{aligned}
$$

Combining (4.184) and (4.185), we deduce that

$$
H(x, D\varphi_r(x)) \leq c\left[H\left(\frac{\cdot}{1-2r}, D\varphi\left(\frac{\cdot}{1-2r}\right)\right) * \psi_r\right](x). \qquad (4.186)
$$

Using the fact that

$$
H\left(\frac{\cdot}{1-2r}, D\varphi\left(\frac{\cdot}{1-2r}\right)\right) * \psi_r \longrightarrow H(\cdot, D\varphi(\cdot)) \quad \text{in} \quad L^1(B_1)
$$

as $r \to 0$, we can apply a generalized version of Lebesgue dominated convergence theorem to obtain a sequence of functions $\{\varphi_k\} := \{\varphi_{r_k}\} \subset C_0^\infty(B_1)$ satisfying (4.183), for a suitable sequence $\{r_k\}$ converging to zero.

We next show that

$$
\int_{B_1} \langle S(x, W), D\varphi_k\rangle\, dx \longrightarrow \int_{B_1} \langle S(x, W), D\varphi\rangle\, dx, \qquad (4.187)
$$

as $k \to \infty$. From the growth condition (4.181) and Young's inequality (2.5) with $\varepsilon = 1$, we have

$$
\begin{aligned}
|\langle S(x, W), D\varphi_k\rangle| &\leq L\left[|W|^{p-1} + a(x)|W|^{p-1}\log(e+|W|)\right]|D\varphi_k| \\
&\leq c\left[H(x, W) + H(x, D\varphi_k)\right],
\end{aligned}
$$

where $c = c(p, L)$ is a positive constant. The generalized version of Lebesgue dominated convergence theorem now yields (4.187). Furthermore, from the fact that

$$
\int_{B_1} \langle S(x, W), D\varphi_k\rangle\, dx = 0, \quad \forall\, k \in \mathbb{N},
$$

we have the desired conclusion (4.182). □

Combining the previous proposition with Remark 4.2.9, we have the following result.

Corollary 4.2.11. *Let $u \in W_0^{1,1}(\Omega)$ be a distributional solution to (4.160) with (4.176), under the assumptions (4.168) on the modulus of continuity $\omega(\cdot)$ of the function $a(\cdot)$. Then we have*

$$\int_\Omega \langle A(x, Du), D\varphi\rangle \, dx = \int_\Omega \langle G(x, F), D\varphi\rangle \, dx,$$

for every function $\varphi \in W_0^{1,1}(\Omega)$ with $H(x, D\varphi) \in L^1(\Omega)$.

Now we prove an existence result for the Dirichlet problem (4.160).

Theorem 4.2.12. *Suppose that $H(x, F) \in L^1(\Omega)$. Then there exists a unique distributional solution $u \in W_0^{1,1}(\Omega)$ to (4.160) such that $H(x, Du) \in L^1(\Omega)$. Furthermore, the standard energy estimate*

$$\int_\Omega H(x, Du) \, dx \le c \int_\Omega H(x, F) \, dx, \tag{4.188}$$

holds for a positive constant $c = c(n, p, \nu, L)$.

Proof. By abuse of notation, we continue to write $H(x, \xi)$ also when $\xi \in \mathbb{R}$. Our proof starts with the observation that the function $H : \Omega \times [0, \infty) \to [0, \infty)$ under consideration is a Musielak-Orlicz function. In addition, $H(\cdot)$ satisfies the Δ_2-condition, as

$$\begin{aligned} H(x, 2t) &= (2t)^p + a(x)(2t)^p \log(e + 2t) \\ &\le 2^{p+1} \left[t^p + a(x)t^p \log(e + t)\right] = 2^{p+1} H(x, t), \end{aligned}$$

for all $x \in \Omega$ and $t \ge 0$. Hence $L^{H(\cdot)}(\Omega)$ is a Banach space. By the absence of Lavrentiev phenomenon discussed in [55] and the result of [116], there exists a distributional solution $u \in W_0^{1,H(\cdot)}(\Omega)$ to the problem (4.160). It follows immediately that $u \in W_0^{1,1}(\Omega)$ with $H(x, Du) \in L^1(\Omega)$. Furthermore, from Corollary 4.2.11, we can choose u as a test function, that is, we have

$$\int_\Omega \langle A(x, Du), Du\rangle \, dx = \int_\Omega \langle G(x, F), Du\rangle \, dx.$$

By using the monotonicity property (4.163) with $\eta = 0$, the growth condition

(4.166) of G and Young's inequality (2.5) with $\tau \in (0,1)$, we obtain

$$\begin{aligned}\widetilde{\nu} \int_\Omega H(x, Du)\, dx &\le \int_\Omega \langle A(x, Du), Du\rangle\, dx = \int_\Omega \langle G(x, F), Du\rangle\, dx \\ &\le L \int_\Omega \left[|F|^{p-1} + a(x)|F|^{p-1} \log(e + |F|)\right] |Du|\, dx \\ &\le L \left[\tau \int_\Omega H(x, Du)\, dx + c(\tau, p) \int_\Omega H(x, F)\, dx\right].\end{aligned}$$

The standard energy estimate (4.188) now follows by taking $\tau = \dfrac{\widetilde{\nu}}{2L}$.

We next show the uniqueness of solutions. Suppose that $u_1, u_2 \in W_0^{1,1}(\Omega)$ are distributional solutions to (4.160) with $H(x, Du_1), H(x, Du_2) \in L^1(\Omega)$. Then we can take $\varphi = u_1 - u_2 \in W_0^{1,1}(\Omega)$ as a test function by Corollary 4.2.11, and hence we obtain

$$\int_\Omega \langle A(x, Du_1) - A(x, Du_2), Du_1 - Du_2\rangle\, dx = 0 \tag{4.189}$$

Combining (4.189) with the monotonicity property (4.163) yields $Du_1 \equiv Du_2$ in Ω. Since $u_1 = 0 = u_2$ on $\partial\Omega$, we conclude that $u_1 \equiv u_2$ in Ω. □

The remainder of this subsection will be devoted to regularity results for reference equations. We first discuss higher integrability results for reference equations in which the nonhomogenous term equals to zero.

Lemma 4.2.13. *Let $h \in W^{1,1}(B_{2r})$ be a distributional solution to*

$$\operatorname{div} A(x, Dh) = 0 \quad \text{in} \quad B_{2r}, \tag{4.190}$$

with $H(x, Dh) \in L^1(B_{2r})$ and $B_{2r} \subset \Omega$. Then there exists a universal constant $\sigma = \sigma(n, p, \nu, L, \omega(\cdot), \|Dh\|_{L^p(B_{2r})}) > 0$ such that

$$\fint_{B_r} [H(x, Dh)]^{1+\sigma}\, dx \le c \left(\fint_{B_{2r}} H(x, Dh)\, dx\right)^{1+\sigma},$$

where $c = c(n, p, \nu, L, \omega(\cdot), \|Dh\|_{L^p(B_{2r})})$ is a positive constant.

Proof. If we prove that the following Caccioppoli type inequality

$$\fint_{B_\rho} H(x, Dh)\, dx \le c \fint_{B_{2\rho}} H\left(x, \frac{|h - (h)_{B_{2\rho}}|}{\rho}\right) dx \tag{4.191}$$

holds whenever $B_{2\rho} \equiv B_{2\rho}(y) \subset B_{2r}$ for some $c = c(n, p, \nu, L) > 0$, then the lemma follows by the same method as in [8, Section 4.1].

To show (4.191), we first define the complementary Musielak-Young function of $H(\cdot)$ by for each $x \in \Omega$,

$$H^*(x,t) := \sup\{ts - H(x,s) : s \ge 0\}.$$

Then $H^* : \Omega \times [0, \infty) \to [0, \infty)$ is also a Musielak-Young function, and it is clear that $H(\cdot) \in \Delta_2 \cap \nabla_2$ and $H^*(\cdot) \in \Delta_2 \cap \nabla_2$. Thus we have the following Young's inequality: for any $\tau \in (0, 1]$, there exists a positive constant $c = c(\tau, p)$ such that for $s, t \ge 0$ and $x \in \Omega$,

$$st \le \tau H^*(x, s) + cH(x, t).$$

It also follows from [101, Lemma 2.2] that there exist $\kappa \in (1, \infty)$ and $c \ge 1$, both depending only on p, such that

$$H^*(x, \theta t) \le c\theta^\kappa H^*(x, t) \tag{4.192}$$

for all $x \in B_{2r}, t \ge 0$ and all $\theta \in [0, 1]$.

Let $q := \frac{\kappa}{\kappa-1} > 1$, and let $\zeta \in C_0^\infty(B_{2\rho})$ be a cut-off function such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in B_ρ and $|D\zeta| \le \frac{2}{\rho}$. Then we note from (4.192) and [101, Lemma 2.2] that

$$H^*\left(x, \frac{H(x,t)}{t}\zeta^{q-1}\right) \le c\,\zeta^q H^*\left(x, \frac{H(x,t)}{t}\right) \le c\,\zeta^q H(x,t) \tag{4.193}$$

for $x \in B_{2\rho}$ and $t > 0$. We now take $\varphi = \zeta^q(h - (h)_{B_{2\rho}})$ as a test function in (4.190). Indeed, Proposition 4.2.10 ensures that the above choice of φ is valid from the fact that $H(x, Dh) \in L^1(B_{2r})$. We thus get

$$\fint_{B_{2\rho}} \langle A(x, Dh), Dh\rangle\, \zeta^q\, dx = -q \fint_{B_{2\rho}} \langle A(x, Dh), D\zeta\rangle\, \zeta^{q-1} \left(h - (h)_{B_{2\rho}}\right) dx.$$

Then it follows from the monotonicity property (4.163) with $\eta = 0$, the growth condition (4.161), Young's inequality and (4.193) that

$$\begin{aligned}
&\widetilde{\nu} \fint_{B_{2\rho}} H(x, Dh)\, \zeta^q \, dx \\
&\le \fint_{B_{2\rho}} \langle A(x, Dh), Dh \rangle \zeta^q \, dx = -q \fint_{B_{2\rho}} \langle A(x, Dh), D\zeta \rangle \zeta^{q-1} \left(h - (h)_{B_{2\rho}}\right) dx \\
&\le 2qL \fint_{B_{2\rho}} \frac{H(x, Dh)}{|Dh|}\, \zeta^{q-1} \frac{\left|h - (h)_{B_{2\rho}}\right|}{\rho}\, dx \\
&\le \tau \fint_{B_{2\rho}} H^* \left(x, \frac{H(x, Dh)}{|Dh|}\, \zeta^{q-1}\right) dx + c(\tau) \fint_{B_{2\rho}} H\left(x, \frac{\left|h - (h)_{B_{2\rho}}\right|}{\rho}\right) dx \\
&\le c\tau \fint_{B_{2\rho}} H(x, Dh)\, \zeta^q \, dx + c(\tau) \fint_{B_{2\rho}} H\left(x, \frac{\left|h - (h)_{B_{2\rho}}\right|}{\rho}\right) dx.
\end{aligned}$$

Consequently, we obtain the Caccioppoli inequality (4.191) by taking $\tau = \dfrac{\widetilde{\nu}}{2c}$, and the lemma follows. $\square$

We remark that the dependence on $\|Dh\|_{L^p(B_{2r})}$ of the constants c and σ in Lemma 4.2.13 is natural when dealing with non-uniformly elliptic problems. Moreover, the constant c is a non-decreasing function of $\|Dh\|_{L^p(B_{2r})}$, as discussed in [8]. Moreover, as in the proof of [101, Theorem 3.9], one can find the boundary version of Lemma 4.2.13 as follows:

Lemma 4.2.14. *Suppose that Ω is (δ, R_0)-Reifenberg flat. Let $h \in W^{1,1}(\Omega_{2r})$ be a distributional solution to*

$$\begin{cases} \operatorname{div} A(x, Dh) = 0 & \text{in} \quad \Omega_{2r}, \\ h = 0 & \text{on} \quad \partial_w \Omega_{2r}, \end{cases}$$

with $H(x, Dh) \in L^1(\Omega_{2r})$, $0 < 2r \le R_0$ and

$$B_{2r}^+ \subset B_{2r} \cap \Omega \subset B_{2r} \cap \{x^n > -4\delta r\}.$$

Then there exists a universal constant $\sigma = \sigma(n, p, \nu, L, \omega(\cdot), \|Dh\|_{L^p(\Omega_{2r})}) > 0$

such that

$$\fint_{\Omega_r} [H(x, Dh)]^{1+\sigma}\, dx \le c \left(\fint_{\Omega_{2r}} H(x, Dh)\, dx \right)^{1+\sigma},$$

where $c = c(n, p, \nu, L, \omega(\cdot), \|Dh\|_{L^p(\Omega_{2r})})$ *is a positive constant.*

We next discuss Lipschitz estimates for reference problems in which the associated nonlinearity has no x-dependence. To be specific, consider a vector-valued function $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following structural conditions:

$$|A_0(\xi)| + |\xi||D_\xi A_0(\xi)| \le L \left[|\xi|^{p-1} + a_0 |\xi|^{p-1} \log(e + |\xi|) \right], \tag{4.194}$$

$$\langle D_\xi A_0(\xi)\eta, \eta\rangle \ge \nu \left[|\xi|^{p-2} + a_0 |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2, \tag{4.195}$$

for every $\xi, \eta \in \mathbb{R}^n$, where $0 < \nu \le L < +\infty$ and $a_0 \ge 0$ are fixed constants. In accordance with (4.174), we use the notation

$$H_0(\xi) := |\xi|^p + a_0 |\xi|^p \log(e + |\xi|), \tag{4.196}$$

for $\xi \in \mathbb{R}^n$.

Then we have the following interior Lipschitz estimate.

Lemma 4.2.15. *[56, 88] Let* $v \in W^{1,1}(B_{2r})$ *be a distributional solution to*

$$\operatorname{div} A_0(Dv) = 0 \quad \text{in} \quad B_{2r},$$

with $H_0(Dv) \in L^1(B_{2r})$. *Then we have* $Dv \in L^\infty(B_r)$ *with the estimate*

$$\sup_{B_r} H_0(Dv) \le c \fint_{B_{2r}} H_0(Dv)\, dx,$$

where $c = (n, p, \nu, L)$ *is a universal constant.*

We next present a boundary version of the above lemma. Note that the mapping $t \mapsto t^p + a_0 t^p \log(e+t)$ is invertible on $[0, \infty)$, to obtain the following result in much the same way as [40, Theorem 2.2].

Lemma 4.2.16. *Let* $v \in W^{1,1}(B_{2r}^+)$ *be a distributional solution to*

$$\begin{cases} \operatorname{div} A_0(Dv) &= 0 \quad \text{in} \quad B_{2r}^+, \\ v &= 0 \quad \text{on} \quad T_{2r}, \end{cases}$$

with $H_0(Dv) \in L^1(B_{2r}^+)$. Then we have $Dv \in L^\infty(B_r^+)$ with the estimate

$$\sup_{B_r^+} H_0(Dv) \leq c \fint_{B_{2r}^+} H_0(Dv)\, dx,$$

where $c = (n, p, \nu, L)$ is a universal constant.

4.2.3 Covering argument

To establish a desired global estimate, we investigate a covering argument. We first assume that

$$0 < \widetilde{R} \leq \min\left\{R_0, \frac{1}{e}\right\}, \tag{4.197}$$

where R_0 is the number given in Theorem 4.2.4. In this subsection and the next subsection, we fix a point $x_0 \in \Omega$ and concentrate on $\Omega_R = \Omega_R(x_0) = B_R(x_0) \cap \Omega$ with $R \leq \widetilde{R}$.

Let $u \in W_0^{1,1}(\Omega)$ be the distributional solution to (4.160) with (4.176). We first select radii r_1, r_2 such that $\frac{R}{2} \leq r_1 < r_2 \leq R$ and consider the upper level sets

$$E(\lambda, s) := \{x \in \Omega_s : H(x, Du(x)) > \lambda\}, \quad \frac{R}{2} \leq s \leq R, \quad \lambda > 0. \tag{4.198}$$

For each $y \in E(\lambda, r_1)$, we define a continuous function $\Phi_y : (0, r_2 - r_1] \to [0, \infty)$ by

$$\Phi_y(\rho) := \fint_{\Omega_\rho(y)} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx, \tag{4.199}$$

where $\delta > 0$ is to be determined later. From the Lebesgue differentiation theorem and (4.198), it follows that for almost every $y \in E(\lambda, r_1)$,

$$\begin{aligned}\lim_{\rho \to 0} \Phi_y(\rho) &= H(y, Du(y)) + \frac{1}{\delta} H(y, F(y)) \\ &\geq H(y, Du(y)) > \lambda. \end{aligned}\tag{4.200}$$

On the other hand, using (4.171), for any $\rho \in \left[\frac{r_2 - r_1}{156}, r_2 - r_1\right]$, we obtain

$$\Phi_y(\rho) = \fint_{\Omega_\rho(y)} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx$$

$$
\begin{aligned}
&\le \frac{|\Omega_{r_2}|}{|\Omega_\rho(y)|} \fint_{\Omega_{r_2}} \left[H(x,Du) + \frac{1}{\delta} H(x,F) \right] dx \\
&\le \frac{|B_{r_2}|}{|B_\rho(y)|} \frac{|B_\rho(y)|}{|\Omega_\rho(y)|} \fint_{\Omega_{r_2}} \left[H(x,Du) + \frac{1}{\delta} H(x,F) \right] dx \\
&\le \left(\frac{r_2}{\frac{r_2-r_1}{156}} \right)^n \left(\frac{16}{7} \right)^n \fint_{\Omega_{r_2}} \left[H(x,Du) + \frac{1}{\delta} H(x,F) \right] dx \\
&\le \frac{400^n r_2^n}{(r_2-r_1)^n} \fint_{\Omega_{r_2}} \left[H(x,Du) + \frac{1}{\delta} H(x,F) \right] dx =: \lambda_0. \qquad (4.201)
\end{aligned}
$$

From now on, we consider positive numbers λ satisfying

$$\lambda > \lambda_0. \qquad (4.202)$$

Since Φ_y is a continuous function, it follows from (4.200)-(4.202) that for almost every $y \in E(\lambda, r_1)$, there exists an exit time radius $\rho_y \in \left(0, \frac{r_2-r_1}{156}\right)$ such that

$$\Phi_y(\rho_y) = \lambda \quad \text{and} \quad \Phi_y(\rho) < \lambda \quad \text{for all } \rho \in (\rho_y, r_2 - r_1]. \qquad (4.203)$$

Note that the family $\{\Omega_{\rho_y}(y)\} = \{B_{\rho_y}(y) \cap \Omega\}$ covers $E(\lambda, r_1)$ up to a negligible set. Applying Vitali's covering lemma, there exists a countable family of disjoint sets $\{\Omega_{\rho_i}(y_i)\}_{i=1}^\infty$ with $y_i \in E(\lambda, r_1)$ and $\rho_i = \rho_{y_i} \in \left(0, \frac{r_2-r_1}{156}\right)$ such that

$$E(\lambda, r_1) \subset \bigcup_{i \ge 1} \Omega_{5\rho_i}(y_i) \cup \text{negligible set}, \qquad (4.204)$$

and

$$\Phi_{y_i}(\rho_i) = \lambda \quad \text{and} \quad \Phi_{y_i}(\rho) < \lambda \quad \text{for all } \rho \in (\rho_i, r_2 - r_1]. \qquad (4.205)$$

We first discuss the interior case $B_{20\rho_i}(y_i) \subset \Omega$. Let us consider the concentric balls

$$B_i^0 \equiv B_{\rho_i}(y_i) = \Omega_{\rho_i}(y_i), \quad B_i^j \equiv B_{5j\rho_i}(y_i) = \Omega_{5j\rho_i}(y_i) \ \text{ for } \ j = 1,2,3,4. \qquad (4.206)$$

We recall that

$$0 < \rho_i < 5j\rho_i \le 20\rho_i < 156\rho_i < r_2 - r_1 \le \frac{R}{2} \le \frac{\widetilde{R}}{2} < \widetilde{R} \le \min\left\{R_0, \frac{1}{e}\right\}. \tag{4.207}$$

Then (4.199) and (4.205) yield that, for $j = 1, 2, 3, 4$,

$$\fint_{B_i^j} \left[H(x, Du) + \frac{1}{\delta} H(x, F)\right] dx < \lambda. \tag{4.208}$$

For the boundary case $B_{20\rho_i}(y_i) \not\subset \Omega$, we fix a boundary point $\widehat{y_i} \in B_{20\rho_i}(y_i) \cap \partial\Omega$. It is clear that

$$\Omega_{5\rho_i}(y_i) \subset \Omega_{25\rho_i}(\widehat{y_i}). \tag{4.209}$$

Since the domain Ω is (δ, R_0)-Reifenberg flat, there exists a coordinate system $\{z^1, \cdots, z^n\}$, after a proper rotation and translation, such that in this new coordinate system,

$$\begin{cases} y_i = z_i, \quad \widehat{y_i} + 156\delta\rho_i(0, \cdots, 0, 1) \text{ is the origin}, \\ B^+_{156\rho_i} \subset \Omega_{156\rho_i} \subset B_{156\rho_i} \cap \{z^n > -312\delta\rho_i\}. \end{cases} \tag{4.210}$$

We assume that

$$0 < \delta \le \frac{1}{312}. \tag{4.211}$$

From (4.209)-(4.211), we see that

$$\Omega_{5\rho_i}(z_i) \subset \Omega_{26\rho_i}(0) \tag{4.212}$$

in the coordinate system. Now we set

$$\Omega_i^0 \equiv \Omega_{\rho_i}(z_i), \quad \Omega_i^j \equiv \Omega_{26j\rho_i} = \Omega_{26j\rho_i}(0) \ \text{ for } \ j = 1, 2, 3, 4, 5. \tag{4.213}$$

Then it follows from (4.209)-(4.213) that

$$\Omega_i^j \subset \Omega_{130\rho_i} \subset \Omega_{156\rho_i}(z_i), \quad B_i^{j+} \equiv B^+_{26j\rho_i} \subset \Omega_i^j \subset B_{26j\rho_i} \cap \{z^n > -312\delta\rho_i\}. \tag{4.214}$$

Hence we deduce that, for $j = 1, 2, 3, 4, 5$,

$$\fint_{\Omega_i^j} \left[H(z, Du) + \frac{1}{\delta} H(z, F)\right] dz$$

$$
\begin{aligned}
&\leq \frac{|\Omega_{156\rho_i}(z_i)|}{|\Omega_{26\rho_i}|} \fint_{\Omega_{156\rho_i}(z_i)} \left[H(z, Du) + \frac{1}{\delta} H(z, F) \right] dz \\
&\leq \frac{|B_{156\rho_i}(z_i)|}{|B^+_{26\rho_i}|} \fint_{\Omega_{156\rho_i}(z_i)} \left[H(z, Du) + \frac{1}{\delta} H(z, F) \right] dz \\
&\leq 2 \cdot 6^n \fint_{\Omega_{156\rho_i}(z_i)} \left[H(z, Du) + \frac{1}{\delta} H(z, F) \right] dz \\
&< 2 \cdot 6^n \lambda. \qquad (4.215)
\end{aligned}
$$

Consequently, we have

$$
\fint_{\Omega_i^5} H(x, Du)\, dx \leq c\lambda \quad \text{and} \quad \fint_{\Omega_i^5} H(x, F)\, dx \leq c\delta\lambda. \qquad (4.216)
$$

4.2.4 Comparison estimates

The comparison estimates will be divided into the interior case and the boundary case. In this subsection, we mainly deal with the boundary case since similar results for the interior case can be proved in much a simpler way. In this case, we adopt the new coordinate system (4.210) and (4.212)-(4.214), as the related quantities are invariant under such translation and rotation.

Let $u \in W_0^{1,1}(\Omega)$ be the distributional solution to (4.160) with (4.176), and let $h_i \in W^{1,1}(\Omega_i^5)$ be the distributional solution to the homogeneous problem

$$
\begin{cases} \operatorname{div} A(x, Dh_i) = 0 & \text{in} \quad \Omega_i^5, \\ h_i = u & \text{on} \quad \partial\Omega_i^5, \end{cases} \qquad (4.217)
$$

with $H(x, Dh_i) \in L^1(\Omega_i^5)$. It follows from the standard energy estimate for (4.217) and the uniform bound (4.216) that

$$
\fint_{\Omega_i^5} H(x, Dh_i)\, dx \leq c \fint_{\Omega_i^5} H(x, Du)\, dx \leq c\lambda, \qquad (4.218)
$$

for a universal constant $c = c(n, p, \nu, L)$.

Lemma 4.2.17. *Let $u \in W_0^{1,1}(\Omega)$ be the distributional solution to (4.160) with (4.176). Then for any $0 < \varepsilon < 1$, there exists a small constant $\delta = \delta(n, p, \nu, L, \varepsilon) > 0$ such that if (4.216) holds and if $h_i \in W^{1,1}(\Omega_i^5)$ is the* *distributional solution to (4.217) with $H(x,Dh_i)\in L^1(\Omega_i^5)$, then we have*

$$\fint_{\Omega_i^5}\left(|V_p(Du)-V_p(Dh_i)|^2+a(x)|V_{\log}(Du)-V_{\log}(Dh_i)|^2\right)dx\le\varepsilon\lambda. \tag{4.219}$$

Proof. We take $\varphi=u-h_i\in W_0^{1,1}(\Omega_i^5)$ as a test function in (4.160) and (4.217) to find that

$$\fint_{\Omega_i^5}\langle A(x,Du)-A(x,Dh_i),D(u-h_i)\rangle\,dx=\fint_{\Omega_i^5}\langle G(x,F),D(u-h_i)\rangle\,dx.$$

Indeed, Corollary 4.2.11 ensures that the above choice of φ is valid from (4.218). Using the monotonicity property (4.163), the growth condition (4.166), Young's inequality (2.5) with $\tau\in(0,1)$, the uniform bounds (4.216) and (4.218), we obtain

$$\begin{aligned}
&\fint_{\Omega_i^5}\left(|V_p(Du)-V_p(Dh_i)|^2+a(x)|V_{\log}(Du)-V_{\log}(Dh_i)|^2\right)dx\\
&\qquad\le c\fint_{\Omega_i^5}\langle A(x,Du)-A(x,Dh_i),D(u-h_i)\rangle\,dx\\
&\qquad= c\fint_{\Omega_i^5}\langle G(x,F),D(u-h_i)\rangle\,dx\\
&\qquad\le c\fint_{\Omega_i^5}\left[|F|^{p-1}+a(x)|F|^{p-1}\log(e+|F|)\right]\left[|Du|+|Dh_i|\right]dx\\
&\qquad\le \tau\fint_{\Omega_i^5}H(x,Du)\,dx+\tau\fint_{\Omega_i^5}H(x,Dh_i)\,dx+c(\tau)\fint_{\Omega_i^5}H(x,F)\,dx\\
&\qquad\le c\tau\lambda+c(\tau)\delta\lambda.
\end{aligned}$$

The conclusion (4.219) now follows by taking $\tau=\dfrac{\varepsilon}{2c}$ and $\delta=\dfrac{\varepsilon}{2c(\tau)}$. □

We next consider a vector field $\widetilde{A}:\Omega_i^4\times\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$\widetilde{A}(x,\xi):=\left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right]A(x,\xi), \tag{4.220}$$

where $x_{i,M} \in \overline{\Omega_i^4}$ is a point such that

$$a(x_{i,M}) = \sup_{x \in \Omega_i^4} a(x). \tag{4.221}$$

This vector field $\widetilde{A}$ actually depends on i, but we omit the index i for simplicity. We now claim that the nonlinearity $\widetilde{A}$ satisfies the following structural conditions: for almost every $x \in \Omega_i^4$ and for all $\xi, \eta \in \mathbb{R}^n$,

$$|\widetilde{A}(x,\xi)| + |\xi||D_\xi \widetilde{A}(x,\xi)| \leq 2L \left[|\xi|^{p-1} + a(x_{i,M})|\xi|^{p-1}\log(e+|\xi|)\right], \tag{4.222}$$

$$\langle D_\xi \widetilde{A}(x,\xi)\eta, \eta\rangle \geq \frac{\nu}{2}\left[|\xi|^{p-2} + a(x_{i,M})|\xi|^{p-2}\log(e+|\xi|)\right]|\eta|^2, \tag{4.223}$$

where L and ν are the positive constants presented in (4.161) and (4.162), respectively. To see this, we first compute $D_\xi \widetilde{A}(x,\xi)$ as follows:

$$\begin{aligned} D_\xi \widetilde{A}(x,\xi) &= \left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right] D_\xi A(x,\xi) \\ &\quad + D_\xi \left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right] \otimes A(x,\xi) \\ &= \frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)} D_\xi A(x,\xi) \\ &\quad + \left[\frac{a(x_{i,M})-a(x)}{(1+a(x)\log(e+|\xi|))^2(e+|\xi|)|\xi|}\right] \xi \otimes A(x,\xi). \end{aligned} \tag{4.224}$$

Then it follows from (4.161), (4.220) and (4.224) that for almost every $x \in \Omega_i^4$ and for all $\xi \in \mathbb{R}^n$,

$$\begin{aligned} &|\widetilde{A}(x,\xi)| + |\xi||D_\xi \widetilde{A}(x,\xi)| \\ &\quad \leq \left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right] (|A(x,\xi)| + |\xi||D_\xi A(x,\xi)|) \\ &\qquad + \left[\frac{a(x_{i,M})-a(x)}{(1+a(x)\log(e+|\xi|))^2} \cdot \frac{|\xi|}{e+|\xi|}\right] |A(x,\xi)| \\ &\quad \leq \left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right] (|A(x,\xi)| + |\xi||D_\xi A(x,\xi)|) \end{aligned}$$

$$
\begin{aligned}
&\quad + \left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right]|A(x,\xi)| \\
&\leq 2\left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right](|A(x,\xi)|+|\xi||D_\xi A(x,\xi)|) \\
&\leq 2L\left[|\xi|^{p-1}+a(x_{i,M})|\xi|^{p-1}\log(e+|\xi|)\right].
\end{aligned}
$$

It also follows from (4.162) and (4.224) that

$$
\begin{aligned}
&\langle D_\xi \widetilde{A}(x,\xi)\eta,\eta\rangle \\
&\quad = \left[\frac{1+a(x_{i,M})\log(e+|\xi|)}{1+a(x)\log(e+|\xi|)}\right]\langle D_\xi A(x,\xi)\eta,\eta\rangle \\
&\qquad + \left[\frac{a(x_{i,M})-a(x)}{(1+a(x)\log(e+|\xi|))^2(e+|\xi|)|\xi|}\right]\langle \xi\otimes A(x,\xi)\eta,\eta\rangle \\
&\quad \geq \nu\left[|\xi|^{p-2}+a(x_{i,M})|\xi|^{p-2}\log(e+|\xi|)\right]|\eta|^2 \\
&\qquad - \left[\frac{a(x_{i,M})-a(x)}{(1+a(x)\log(e+|\xi|))^2(e+|\xi|)}\right]|A(x,\xi)||\eta|^2 \\
&\quad \geq \nu\left[|\xi|^{p-2}+a(x_{i,M})|\xi|^{p-2}\log(e+|\xi|)\right]|\eta|^2 \\
&\qquad - L\left[a(x_{i,M})-a(x)\right]\left[\frac{|\xi|^{p-1}+a(x)|\xi|^{p-1}\log(e+|\xi|)}{(1+a(x)\log(e+|\xi|))^2(e+|\xi|)}\right]|\eta|^2 \\
&\quad \geq \nu\left[|\xi|^{p-2}+a(x_{i,M})|\xi|^{p-2}\log(e+|\xi|)\right]|\eta|^2 \\
&\qquad - L\left[a(x_{i,M})-a(x)\right]|\xi|^{p-2}|\eta|^2 \\
&\quad \geq \left(\nu - L\left[a(x_{i,M})-a(x)\right]\right)\left[|\xi|^{p-2}+a(x_{i,M})|\xi|^{p-2}\log(e+|\xi|)\right]|\eta|^2.
\end{aligned}
$$

We now choose $\widetilde{R}>0$ so small that

$$
0 \leq a(x_{i,M})-a(x) \leq \omega(208\rho_i) \leq \omega(\widetilde{R}) \leq \frac{\nu}{2L}. \tag{4.225}
$$

Then the structural condition (4.223) follows.

We remark that the structural condition (4.223) implies the following monotonicity property: for almost every $x\in\Omega_i^4$ and for all $\xi,\eta\in\mathbb{R}^n$,

$$
\langle \widetilde{A}(x,\xi)-\widetilde{A}(x,\eta),\xi-\eta\rangle \geq \frac{\nu}{2}\left[|V_p(\xi)-V_p(\eta)|^2+a(x_{i,M})|V_{\log}(\xi)-V_{\log}(\eta)|^2\right]. \tag{4.226}
$$

With the solution h_i to (4.217), we next let w_i be the distributional solution to

$$\begin{cases} \operatorname{div} \widetilde{A}(x, Dw_i) & = & 0 & \text{in} & \Omega_i^4, \\ w_i & = & h_i & \text{on} & \partial\Omega_i^4. \end{cases} \tag{4.227}$$

According to Lemma 4.2.14, we see that $Dh_i \in L^{p(1+\sigma)}(\Omega_i^4)$ for some positive constant $\sigma = \sigma(n, p, \nu, L, \omega(\cdot), \|Dh_i\|_{L^p(\Omega_i^4)}) > 0$. From the fact $L^{p(1+\sigma)}(\Omega_i^4) \subset L^p \log L(\Omega_i^4)$, we have

$$\int_{\Omega_i^4} H(x_{i,M}, Dh_i)\, dx = \int_{\Omega_i^4} |Dh_i|^p\, dx + a(x_{i,M}) \int_{\Omega_i^4} |Dh_i|^p \log(e+|Dh_i|)\, dx < +\infty.$$

In addition, choosing $w_i - h_i$ as a test function in (4.227) and using the structural conditions (4.222)-(4.223), we have the standard energy estimate for (4.227) as follows:

$$\fint_{\Omega_i^4} H(x_{i,M}, Dw_i)\, dx \le c \fint_{\Omega_i^4} H(x_{i,M}, Dh_i)\, dx < +\infty, \tag{4.228}$$

for a universal constant $c = c(n, p, \nu, L)$.

Lemma 4.2.18. *Under the assumptions and conclusions in Lemma 4.2.17, there exists $\widetilde{R} = \widetilde{R}(n, p, \nu, L, \omega(\cdot), \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$ such that if $w_i \in W^{1,1}(\Omega_i^4)$ is the distributional solution to (4.227) with $H(x_{i,M}, Dw_i) \in L^1(\Omega_i^4)$, then we have*

$$\fint_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \le \varepsilon\lambda. \tag{4.229}$$

Furthermore, we have

$$\fint_{\Omega_i^4} H(x_{i,M}, Dh_i)\, dx \le c\lambda, \tag{4.230}$$

where $c = c(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ is a positive constant.

Proof. We take $\varphi = h_i - w_i \in W_0^{1,1}(\Omega_i^4)$ as a test function in (4.217) and (4.227). Indeed, Proposition 4.2.10 and Remark 4.2.9 ensure that the above choice of φ is valid from (4.221) and (4.228), and hence we have

$$\fint_{\Omega_i^4} \langle \widetilde{A}(x, Dh_i) - \widetilde{A}(x, Dw_i), D(h_i - w_i)\rangle\, dx$$

$$= \fint_{\Omega_i^4} \langle \widetilde{A}(x, Dh_i) - A(x, Dh_i), D(h_i - w_i) \rangle \, dx. \quad (4.231)$$

It follows from (4.226) and (4.231) that

$$\begin{aligned}
&\fint_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
&\qquad \le c \fint_{\Omega_i^4} \langle \widetilde{A}(x, Dh_i) - \widetilde{A}(x, Dw_i), D(h_i - w_i) \rangle \, dx \\
&\qquad = c \fint_{\Omega_i^4} \langle \widetilde{A}(x, Dh_i) - A(x, Dh_i), D(h_i - w_i) \rangle \, dx \\
&\qquad \le c \fint_{\Omega_i^4} |\widetilde{A}(x, Dh_i) - A(x, Dh_i)| |Dh_i - Dw_i| \, dx. \qquad (4.232)
\end{aligned}$$

By the definition of $\widetilde{A}$ and the growth condition of A, we have

$$\begin{aligned}
|\widetilde{A}(x, Dh_i) - A(x, Dh_i)| &= \left[\frac{1 + a(x_{i,M}) \log(e + |Dh_i|)}{1 + a(x) \log(e + |Dh_i|)} - 1 \right] |A(x, Dh_i)| \\
&= \left[\frac{(a(x_{i,M}) - a(x)) \log(e + |Dh_i|)}{1 + a(x) \log(e + |Dh_i|)} \right] |A(x, Dh_i)| \\
&\le L(a(x_{i,M}) - a(x)) |Dh_i|^{p-1} \log(e + |Dh_i|) \\
&\le L\omega(208\rho_i) |Dh_i|^{p-1} \log(e + |Dh_i|). \qquad (4.233)
\end{aligned}$$

Note that $\omega(c\rho_i) \le c\omega(\rho_i)$ for $c \ge 1$, by the concavity of $\omega(\cdot)$. Then it follows from (4.232), (4.233) and Young's inequality with $\tau \in (0, 1)$ that

$$\begin{aligned}
&\fint_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
&\qquad \le c \fint_{\Omega_i^4} \omega(\rho_i) |Dh_i|^{p-1} \log(e + |Dh_i|)(|Dh_i| + |Dw_i|) \, dx \\
&\qquad \le c\tau^{-\frac{1}{p-1}} \fint_{\Omega_i^4} \omega(\rho_i)^{\frac{p}{p-1}} |Dh_i|^p \log^{\frac{p}{p-1}}(e + |Dh_i|) \, dx \\
&\qquad\quad + c\tau \fint_{\Omega_i^4} (|Dh_i| + |Dw_i|)^p \, dx
\end{aligned}$$

$$
\begin{aligned}
&\leq c_1\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\fint_{\Omega_i^4}|Dh_i|^p\log^{\frac{p}{p-1}}(e+|Dh_i|)\,dx\\
&\qquad+c_1\tau\fint_{\Omega_i^4}(|Dh_i|^p+|Dw_i|^p)\,dx\\
&=:\mathrm{I}+\mathrm{II},
\end{aligned}\tag{4.234}
$$

for some positive constant $c_1=c_1(n,p,\nu,L)$.

We first estimate I. Using (2.10), (2.11), (4.207) and Lemma 4.2.14, we derive

$$
\begin{aligned}
\mathrm{I}&=c_1\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\fint_{\Omega_i^4}|Dh_i|^p\log^{\frac{p}{p-1}}(e+|Dh_i|)\,dx\\
&\leq c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\fint_{\Omega_i^4}|Dh_i|^p\log^{\frac{p}{p-1}}\left(e+\frac{|Dh_i|^p}{(|Dh_i|^p)_{\Omega_i^4}}\right)dx\\
&\qquad+c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\log^{\frac{p}{p-1}}\left(e+(|Dh_i|^p)_{\Omega_i^4}\right)\fint_{\Omega_i^4}|Dh_i|^p\,dx\\
&\leq c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\left(\fint_{\Omega_i^4}|Dh_i|^{p(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}}\\
&\qquad+c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\log^{\frac{p}{p-1}}\left(e+\frac{c}{\rho_i^n}\int_{\Omega_i^4}H(x,Dh_i)\,dx\right)\fint_{\Omega_i^4}|Dh_i|^p\,dx\\
&\leq c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\left(\fint_{\Omega_i^4}[H(x,Dh_i)]^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}}\\
&\qquad+c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\log^{\frac{p}{p-1}}\left(e+\frac{c}{\rho_i^n}\left\|H(\cdot,F)\right\|_{L^1(\Omega)}\right)\fint_{\Omega_i^4}H(x,Dh_i)\,dx\\
&\leq c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\fint_{\Omega_i^5}H(x,Dh_i)\,dx\\
&\qquad+c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\log^{\frac{p}{p-1}}\left(\frac{1}{\rho_i}\right)\fint_{\Omega_i^4}H(x,Dh_i)\,dx\\
&\leq c\tau^{-\frac{1}{p-1}}\omega(\rho_i)^{\frac{p}{p-1}}\log^{\frac{p}{p-1}}\left(\frac{1}{\rho_i}\right)\fint_{\Omega_i^5}H(x,Dh_i)\,dx\\
&\leq c\tau^{-\frac{1}{p-1}}\left(\omega(\rho_i)\log\left(\frac{1}{\rho_i}\right)\right)^{\frac{p}{p-1}}\lambda,
\end{aligned}\tag{4.235}
$$

for a positive constant $c = c(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$. In the same manner we see that

$$\omega(\rho_i) \fint_{\Omega_i^4} |Dh_i|^p \log(e + |Dh_i|) \, dx \leq c\omega(\rho_i) \log\left(\frac{1}{\rho_i}\right) \fint_{\Omega_i^5} H(x, Dh_i) \, dx,$$

and hence

$$\begin{aligned}
\fint_{\Omega_i^4} H(x_{i,M}, Dh_i) \, dx &= \fint_{\Omega_i^4} H(x, Dh_i) \, dx \\
&\quad + \fint_{\Omega_i^4} (a(x_{i,M}) - a(x)) |Dh_i|^p \log(e + |Dh_i|) \, dx \\
&\leq \fint_{\Omega_i^4} H(x, Dh_i) \, dx + c\omega(\rho_i) \fint_{\Omega_i^4} |Dh_i|^p \log(e + |Dh_i|) \, dx \\
&\leq c \left(1 + \omega(\rho_i) \log\left(\frac{1}{\rho_i}\right)\right) \fint_{\Omega_i^5} H(x, Dh_i) \, dx \\
&\leq c \fint_{\Omega_i^5} H(x, Dh_i) \, dx. \qquad (4.236)
\end{aligned}$$

Now the conclusion (4.230) follows from (4.236) and (4.218).

We next estimate II. Using (4.228), (4.236) and (4.218), we have

$$\begin{aligned}
\mathrm{II} &= c_1 \tau \fint_{\Omega_i^4} (|Dh_i|^p + |Dw_i|^p) \, dx \\
&\leq c\tau \fint_{\Omega_i^4} \left[H(x_{i,M}, Dh_i) + H(x_{i,M}, Dw_i)\right] \, dx \\
&\leq c\tau \fint_{\Omega_i^4} H(x_{i,M}, Dh_i) \, dx \leq c\tau \fint_{\Omega_i^5} H(x, Dh_i) \, dx \leq c\tau\lambda. \qquad (4.237)
\end{aligned}$$

Combining (4.234) with (4.235) and (4.237) yields

$$\begin{aligned}
&\fint_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\
&\qquad \leq c \left[\tau^{-\frac{1}{p-1}} \left(\omega(\rho_i) \log\left(\frac{1}{\rho_i}\right)\right)^{\frac{p}{p-1}} + \tau\right] \lambda.
\end{aligned}$$

Taking $\tau = \omega(\rho_i)\log\left(\frac{1}{\rho_i}\right)$ in the above inequality, we get

$$\begin{aligned}\fint_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\ \leq c_2\omega(\rho_i)\log\left(\frac{1}{\rho_i}\right)\lambda, \quad (4.238)\end{aligned}$$

for some positive constant $c_2 = c_2(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$. Now, from (4.168), we take so small $\widetilde{R} > 0$ that for any $r \leq \widetilde{R}$,

$$c_2\omega(r)\log\left(\frac{1}{r}\right) \leq \varepsilon. \tag{4.239}$$

Then the conclusion (4.229) follows from (4.207), (4.238) and (4.239). $\square$

Let $\widetilde{A}_0(\xi)$ denote the integral average of $\widetilde{A}(\cdot, \xi)$ over B_i^{3+} with respect to x-variable for each fixed $\xi \in \mathbb{R}^n$, that is,

$$\widetilde{A}_0(\xi) = \fint_{B_i^{3+}} \widetilde{A}(x, \xi)\, dx = \frac{1}{|B_i^{3+}|}\int_{B_i^{3+}} \widetilde{A}(x, \xi)\, dx. \tag{4.240}$$

We note that $\widetilde{A}_0(\xi)$ satisfies the structural conditions (4.222) and (4.223). We now let v_i be the unique distributional solution to

$$\begin{cases} \operatorname{div} \widetilde{A}_0(Dv_i) &= 0 \quad \text{in} \quad \Omega_i^3, \\ v_i &= w_i \quad \text{on} \quad \partial\Omega_i^3. \end{cases} \tag{4.241}$$

with $H(x_{i,M}, Dv_i) \in L^1(\Omega_i^3)$. It follows from (4.228), (4.230) and the standard energy estimate for (4.241) with the choice of $a(\cdot) \equiv a(x_{i,M})$ that

$$\fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\, dx \leq c\fint_{\Omega_i^3} H(x_{i,M}, Dw_i)\, dx \leq c\lambda, \tag{4.242}$$

for a positive constant $c = c(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$.

Lemma 4.2.19. *Let $u \in W_0^{1,1}(\Omega)$ be the distributional solution to (4.160) with (4.176). Then for any $0 < \varepsilon < 1$, there exists a small constant $\delta = \delta(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$ such that if (4.216) holds and if $h_i \in$*

$W^{1,1}(\Omega_i^5)$ is the distributional solution to (4.217) with $H(x, Dh_i) \in L^1(\Omega_i^5)$, $w_i \in W^{1,1}(\Omega_i^4)$ is the distributional solution to (4.227) with $H(x_{i,M}, Dw_i) \in L^1(\Omega_i^4)$ and $v_i \in W^{1,1}(\Omega_i^3)$ is the distributional solution to (4.241) with $H(x_{i,M}, Dv_i) \in L^1(\Omega_i^3)$, then we have

$$\fint_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \leq \varepsilon\lambda. \tag{4.243}$$

Proof. We take $\varphi = w_i - v_i \in W_0^{1,1}(\Omega_i^3)$ as a test function in (4.227) and (4.241) to see that

$$\begin{aligned} &\fint_{\Omega_i^3} \langle \widetilde{A}_0(Dw_i) - \widetilde{A}_0(Dv_i), D(w_i - v_i) \rangle \, dx \\ &\qquad = - \fint_{\Omega_i^3} \langle \widetilde{A}(x, Dw_i) - \widetilde{A}_0(Dw_i), D(w_i - v_i) \rangle \, dx. \end{aligned} \tag{4.244}$$

It follows from (4.226) and (4.244) that

$$\begin{aligned} &\fint_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\ &\quad \leq c \fint_{\Omega_i^3} \langle \widetilde{A}_0(Dw_i) - \widetilde{A}_0(Dv_i), D(w_i - v_i) \rangle \, dx \\ &\quad = -c \fint_{\Omega_i^3} \langle \widetilde{A}(x, Dw_i) - \widetilde{A}_0(Dw_i), D(w_i - v_i) \rangle \, dx \\ &\quad \leq c \fint_{\Omega_i^3} |\widetilde{A}(x, Dw_i) - \widetilde{A}_0(Dw_i)| |Dw_i - Dv_i| \, dx. \end{aligned} \tag{4.245}$$

From a direct calculation, we obtain

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\widetilde{A}(x, \xi) - \widetilde{A}_0(\xi)|}{|\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|)} = \Theta(A, B_i^{3+})(x). \tag{4.246}$$

Indeed, it follows from the definition of $\widetilde{A}(x, \xi)$ that

$$\frac{|\widetilde{A}(x, \xi) - \widetilde{A}_0(\xi)|}{|\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|)}$$

$$
\begin{aligned}
&= \left| \frac{\widetilde{A}(x,\xi)}{|\xi|^{p-1} + a(x_{i,M})|\xi|^{p-1}\log(e+|\xi|)} - \frac{\fint_{B_i^{3+}} \widetilde{A}(x,\xi)\,dx}{|\xi|^{p-1} + a(x_{i,M})|\xi|^{p-1}\log(e+|\xi|)} \right| \\
&= \left| \frac{A(x,\xi)}{|\xi|^{p-1} + a(x)|\xi|^{p-1}\log(e+|\xi|)} - \fint_{B_i^{3+}} \frac{A(x,\xi)}{|\xi|^{p-1} + a(x)|\xi|^{p-1}\log(e+|\xi|)}\,dx \right| \\
&= \left| \frac{A(x,\xi)}{|\xi|^{p-1} + a(x)|\xi|^{p-1}\log(e+|\xi|)} - \left(\frac{A(\cdot,\xi)}{|\xi|^{p-1} + a(\cdot)|\xi|^{p-1}\log(e+|\xi|)} \right)_{B_i^{3+}} \right|,
\end{aligned}
$$

and hence the equality (4.246) follows from (4.169). Substituting (4.246) into (4.245), we have

$$
\begin{aligned}
&\fint_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\
&\le c \fint_{\Omega_i^3} \Theta(A, B_i^{3+}) \left[|Dw_i|^{p-1} + a(x_{i,M})|Dw_i|^{p-1}\log(e+|Dw_i|) \right] |Dw_i - Dv_i|\,dx \\
&\le c_0 \fint_{\Omega_i^3} \Theta(A, B_i^{3+}) \left[|Dw_i|^{p} + a(x_{i,M})|Dw_i|^{p}\log(e+|Dw_i|) \right] dx \\
&\qquad + c_0 \fint_{\Omega_i^3} \Theta(A, B_i^{3+}) |Dw_i|^{p-1}|Dv_i|\,dx \\
&\qquad + c_0 \fint_{\Omega_i^3} \Theta(A, B_i^{3+})\, a(x_{i,M})|Dw_i|^{p-1}\log(e+|Dw_i|)|Dv_i|\,dx \\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,
\end{aligned} \tag{4.247}
$$

for some positive constant $c_0 = c_0(n,p,\nu,L)$.

We first estimate I_1. Using Hölder's inequality and the higher integrability for Dw_i with the choice of $a(\cdot) \equiv a(x_{i,M})$, we get

$$
\begin{aligned}
\mathrm{I}_1 &= c_0 \fint_{\Omega_i^3} \Theta(A, B_i^{3+})\, H(x_{i,M}, Dw_i)\,dx \\
&\le c \left(\fint_{\Omega_i^3} \Theta^{\frac{1+\sigma}{\sigma}}\,dx \right)^{\frac{\sigma}{1+\sigma}} \left(\fint_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma}\,dx \right)^{\frac{1}{1+\sigma}} \\
&\le c \left(\fint_{\Omega_i^3} \Theta^{\frac{1+\sigma}{\sigma}}\,dx \right)^{\frac{\sigma}{1+\sigma}} \fint_{\Omega_i^4} H(x_{i,M}, Dw_i)\,dx.
\end{aligned}
$$

We note from (4.170) and (4.214) that

$$\begin{aligned}
\fint_{\Omega_i^3} \Theta^{\frac{1+\sigma}{\sigma}}\,dx &\le \frac{1}{|B_i^{3+}|}\left(\int_{B_i^{3+}} \Theta^{\frac{1+\sigma}{\sigma}}\,dx + \int_{\Omega_i^3\setminus B_i^{3+}} \Theta^{\frac{1+\sigma}{\sigma}}\,dx\right) \\
&\le \fint_{B_i^{3+}} \Theta^{\frac{1+\sigma}{\sigma}}\,dx + \frac{|\Omega_i^3\setminus B_i^{3+}|}{|B_i^{3+}|}(2L)^{\frac{1+\sigma}{\sigma}} \\
&\le c\delta. \qquad (4.248)
\end{aligned}$$

Using the energy estimates (4.228) and (4.230), we get

$$\mathrm{I}_1 \le c\delta^{\frac{\sigma}{1+\sigma}} \fint_{\Omega_i^4} H(x_{i,M}, Dw_i)\,dx \le c\delta^{\frac{\sigma}{1+\sigma}}\lambda. \qquad (4.249)$$

We next estimate I_2. Using Young's inequality with $\tau \in (0,1)$, Hölder's inequality and the higher integrability for Dw_i, we derive

$$\begin{aligned}
\mathrm{I}_2 &= c_0 \fint_{\Omega_i^3} \Theta(A, B_i^{3+})|Dw_i|^{p-1}|Dv_i|\,dx \\
&\le c\tau^{-\frac{1}{p-1}} \fint_{\Omega_i^3} \Theta^{\frac{p}{p-1}}|Dw_i|^p\,dx + c\tau \fint_{\Omega_i^3} |Dv_i|^p\,dx \\
&\le c\tau^{-\frac{1}{p-1}} \fint_{\Omega_i^3} \Theta^{\frac{p}{p-1}} H(x_{i,M}, Dw_i)\,dx + c\tau \fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\,dx \\
&\le c\tau^{-\frac{1}{p-1}} \left(\fint_{\Omega_i^3} \Theta^{\frac{p}{p-1}\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{\sigma}{1+\sigma}} \left(\fint_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} \\
&\quad + c\tau \fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\,dx \\
&\le c\tau^{-\frac{1}{p-1}} \left(\fint_{\Omega_i^3} \Theta^{\frac{p}{p-1}\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{\sigma}{1+\sigma}} \fint_{\Omega_i^4} H(x_{i,M}, Dw_i)\,dx \\
&\quad + c\tau \fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\,dx.
\end{aligned}$$

Likewise as in (4.248), one see that

$$\fint_{\Omega_i^3} \Theta^{\frac{p}{p-1}\frac{1+\sigma}{\sigma}}\,dx \le c\delta.$$

Utilizing the energy estimates (4.228), (4.230) and (4.242), we have

$$\begin{aligned} \mathrm{I}_2 &\le c\tau^{-\frac{1}{p-1}}\delta^{\frac{\sigma}{1+\sigma}}\fint_{\Omega_i^4} H(x_{i,M}, Dw_i)\,dx + c\tau\fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\,dx \\ &\le c\tau^{-\frac{1}{p-1}}\delta^{\frac{\sigma}{1+\sigma}}\lambda + c\tau\lambda. \end{aligned} \tag{4.250}$$

We now estimate I_3. To do this, we set $g(t) := t^p\log(e+t)$ for $t \ge 0$. Note that $g \in \Delta_2 \cap \nabla_2$. Using Young's inequality (2.2) with $\tau \in (0,1)$, (2.1) and (2.3), we have

$$\begin{aligned} \mathrm{I}_3 &= c_0 a(x_{i,M})\fint_{\Omega_i^3} \Theta(A, B_i^{3+})|Dw_i|^{p-1}\log(e+|Dw_i|)|Dv_i|\,dx \\ &\le c(\tau)a(x_{i,M})\fint_{\Omega_i^3} g^*\left(\Theta\,\frac{g(|Dw_i|)}{|Dw_i|}\right)dx + c\tau a(x_{i,M})\fint_{\Omega_i^3} g(|Dv_i|)\,dx \\ &\le c(\tau)a(x_{i,M})\fint_{\Omega_i^3} \max\{\Theta^{\kappa_1}, \Theta^{\kappa_2}\}\, g^*\left(\frac{g(|Dw_i|)}{|Dw_i|}\right)dx \\ &\quad + c\tau a(x_{i,M})\fint_{\Omega_i^3} g(|Dv_i|)\,dx \\ &\le c(\tau)\fint_{\Omega_i^3} \left(\Theta^{\kappa_1} + \Theta^{\kappa_2}\right) a(x_{i,M})g(|Dw_i|)\,dx + c\tau\fint_{\Omega_i^3} a(x_{i,M})g(|Dv_i|)\,dx \\ &\le c(\tau)\fint_{\Omega_i^3} \left(\Theta^{\kappa_1} + \Theta^{\kappa_2}\right) H(x_{i,M}, Dw_i)\,dx + c\tau\fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\,dx, \end{aligned}$$

for some positive constants κ_1 and κ_2 depending only on p. Then it follows from the higher integrability for Dw_i that

$$\begin{aligned} \mathrm{I}_3 &\le c(\tau)\left(\fint_{\Omega_i^3} \Theta^{\kappa_1\cdot\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{\sigma}{1+\sigma}} \left(\fint_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} \\ &\quad + c(\tau)\left(\fint_{\Omega_i^3} \Theta^{\kappa_2\cdot\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{\sigma}{1+\sigma}} \left(\fint_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} \end{aligned}$$

$$
\begin{aligned}
&+ c\tau \fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\,dx \\
&\le c(\tau)\left(\fint_{\Omega_i^3} \Theta^{\kappa_1\cdot\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{\sigma}{1+\sigma}} \fint_{\Omega_i^4} H(x_{i,M}, Dw_i)\,dx \\
&\quad + c(\tau)\left(\fint_{\Omega_i^3} \Theta^{\kappa_2\cdot\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{\sigma}{1+\sigma}} \fint_{\Omega_i^4} H(x_{i,M}, Dw_i)\,dx \\
&\quad + c\tau \fint_{\Omega_i^3} H(x_{i,M}, Dv_i)\,dx.
\end{aligned}
$$

By a similar argument, we obtain

$$
\mathrm{I}_3 \le c(\tau)\delta^{\frac{\sigma}{1+\sigma}}\lambda + c\tau\lambda. \tag{4.251}
$$

Combining (4.247) with (4.249)-(4.251) yields

$$
\begin{aligned}
&\fint_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \\
&\quad \le c\delta^{\frac{\sigma}{1+\sigma}}\lambda + c\tau^{-\frac{1}{p-1}}\delta^{\frac{\sigma}{1+\sigma}}\lambda + c(\tau)\delta^{\frac{\sigma}{1+\sigma}}\lambda + c\tau\lambda \\
&\quad \le c(\tau)\delta^{\frac{\sigma}{1+\sigma}}\lambda + c\tau\lambda.
\end{aligned}
$$

The conclusion (4.243) now follows by taking $\tau = \dfrac{\varepsilon}{2c}$ and $\delta = \left(\dfrac{\varepsilon}{2c(\tau)}\right)^{\frac{1+\sigma}{\sigma}}$.
$\square$

Since the domain under consideration can go beyond Lipschitz category, a boundary issue arises and we should deal with it more carefully. The following compactness lemma is the key to get a desired comparison estimate for the boundary case.

Lemma 4.2.20. *Let $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function satisfying the structural conditions (4.194) and (4.195). Then for any $\varepsilon > 0$, there exists a small $\delta = \delta(n, p, L, a_0, \varepsilon) > 0$ such that if*

$$
B_{3r}^{+} \subset \Omega_{3r} \subset B_{3r} \cap \{x^n > -6\delta r\}, \tag{4.252}
$$

for some fixed $r > 0$, and if $v \in W^{1,1}(\Omega_{3r})$ is a distributional solution to

$$\begin{cases} \operatorname{div} A_0(Dv) &= 0 \quad \text{in} \quad \Omega_{3r}, \\ v &= 0 \quad \text{on} \quad \partial_w \Omega_{3r}, \end{cases} \tag{4.253}$$

with $H_0(Dv) \in L^1(\Omega_{3r})$ and

$$\fint_{\Omega_{3r}} H_0(Dv)\,dx \leq \widetilde{c} \tag{4.254}$$

for some constant $\widetilde{c} > 1$, then there exists a distributional solution $\overline{v} \in W^{1,1}(B_{2r}^+)$ to

$$\begin{cases} \operatorname{div} A_0(D\overline{v}) &= 0 \quad \text{in} \quad B_{2r}^+, \\ \overline{v} &= 0 \quad \text{on} \quad T_{2r}, \end{cases} \tag{4.255}$$

with

$$\fint_{B_{2r}^+} H_0(D\overline{v})\,dx \leq \left(\frac{3}{2}\right)^n \widetilde{c}, \tag{4.256}$$

such that

$$\fint_{\Omega_r} \left(|V_p(Dv) - V_p(D\overline{v})|^2 + a_0|V_{\log}(Dv) - V_{\log}(D\overline{v})|^2\right) dx \leq \widetilde{c}\varepsilon, \tag{4.257}$$

where $\overline{v}$ is extended by zero from B_r^+ to $B_r \supset \Omega_r$.

Proof. We first note that it suffices to prove the lemma only for the case $r = 1$ by scaling. Indeed, if we set $\widehat{v}(x) := \frac{1}{r}v(rx)$ for $x \in \widehat{\Omega}_3$, where $\widehat{\Omega}_3 := \left\{\frac{1}{r}x : x \in \Omega_{3r}\right\}$, then for any $x \in \widehat{\Omega}_3$, we see that $D\widehat{v}(x) = Dv(rx)$,

$$\begin{aligned} H_0(D\widehat{v})(x) &= |D\widehat{v}(x)|^p + a_0|D\widehat{v}(x)|^p \log(e + |D\widehat{v}(x)|) \\ &= |Dv(rx)|^p + a_0|Dv(rx)|^p \log(e + |Dv(rx)|) = H_0(Dv)(rx), \end{aligned}$$

$$B_3^+ \subset \widehat{\Omega}_3 \subset B_3 \cap \{x^n > -6\delta\}$$

and $\widehat{v} \in W^{1,1}(\widehat{\Omega}_3)$ is a distributional solution to

$$\begin{cases} \operatorname{div} A_0(D\widehat{v}) &= 0 \quad \text{in} \quad \widehat{\Omega}_3, \\ \widehat{v} &= 0 \quad \text{on} \quad \partial_w \widehat{\Omega}_3, \end{cases}$$

with

$$\fint_{\widehat{\Omega}_3} H_0(D\widehat{v})\,dx \le \widetilde{c}.$$

In the same manner, one can regain (4.255)-(4.257).

We prove this lemma by contradiction via a compactness argument. If not, then we could find $\varepsilon_0 > 0$, $\{\Omega_3^k\}_{k=1}^{\infty}$ with

$$B_3^+ \subset \Omega_3^k \subset B_3 \cap \left\{x^n > -\frac{6}{k}\right\}, \tag{4.258}$$

and distributional solutions $v_k \in W^{1,1}(\Omega_3^k)$ to

$$\begin{cases} \operatorname{div} A_0(Dv_k) &= 0 \quad \text{in} \quad \Omega_3^k, \\ v_k &= 0 \quad \text{on} \quad \partial_w \Omega_3^k, \end{cases} \tag{4.259}$$

with

$$\fint_{\Omega_3^k} H_0(Dv_k)\,dx \le \widetilde{c} \tag{4.260}$$

such that

$$\fint_{\Omega_1^k} \left(|V_p(Dv_k) - V_p(D\overline{v})|^2 + a_0|V_{\log}(Dv_k) - V_{\log}(D\overline{v})|^2\right) dx > \widetilde{c}\varepsilon_0, \tag{4.261}$$

for any distributional solution $\overline{v} \in W^{1,1}(B_2^+)$ to (4.255) with (4.256) when $r = 1$.

By abuse of notation, we shall continue to write $H_0(\xi)$ also when $\xi \in \mathbb{R}$. Then the function $H_0(t) = t^p + a_0 t^p \log(e+t)$, $t \ge 0$ is a Young function with $H_0 \in \Delta_2 \cap \nabla_2$. Let us consider the Orlicz space $L^{H_0}(\Omega)$. We remark that the space $L^{H_0}(\Omega)$ is nothing but the Lebesgue space $L^p(\Omega)$ if $a_0 = 0$, and the Orlicz space $L^p \log L(\Omega)$ if $a_0 > 0$. Therefore, the space $L^{H_0}(\Omega)$ and the associated Sobolev space $W^{1,H_0}(\Omega)$ are reflexive Banach spaces.

Let $\widetilde{v_k}$ be the zero extension of v_k from Ω_3^k to B_3. Then by Lemma 4.2.7 and (4.260), $\{H_0(D\widetilde{v_k})\}_{k=1}^{\infty}$ is uniformly bounded in $L^1(B_3)$. Hence $\{\widetilde{v_k}\}_{k=1}^{\infty}$ is uniformly bounded in $W^{1,H_0}(B_3)$ and also $\{v_k\}_{k=1}^{\infty}$ is uniformly bounded in $W^{1,H_0}(B_3^+)$. From the above remark, there exist subsequences, which we still denote by $\{\widetilde{v_k}\}_{k=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$, $v_0 \in W^{1,H_0}(B_3)$ and $v_\infty \in W^{1,H_0}(B_3^+)$ such that

$$\widetilde{v_k} \rightharpoonup v_0 \text{ weakly in } W^{1,H_0}(B_3), \quad \widetilde{v_k} \to v_0 \text{ strongly in } L^{H_0}(B_3), \tag{4.262}$$

and

$$v_k \rightharpoonup v_\infty \text{ weakly in } W^{1,H_0}(B_3^+), \quad v_k \to v_\infty \text{ strongly in } L^{H_0}(B_3^+). \tag{4.263}$$

We note from (4.258), (4.259), (4.260) and (4.263) that v_∞ is a solution to (4.255) when $r = 1$, see [29]. Furthermore, it follows from the weakly lower semicontinuity, (4.258), (4.260) and (4.263) that

$$\begin{aligned} \fint_{B_2^+} H_0(Dv_\infty)\,dx &\le \left(\frac{3}{2}\right)^n \fint_{B_3^+} H_0(Dv_\infty)\,dx \\ &\le \left(\frac{3}{2}\right)^n \liminf_{k\to\infty} \fint_{B_3^+} H_0(Dv_k)\,dx \\ &= \left(\frac{3}{2}\right)^n \liminf_{k\to\infty} \frac{|\Omega_3^k|}{|B_3^+|} \fint_{\Omega_3^k} H_0(Dv_k)\,dx \le \left(\frac{3}{2}\right)^n \widetilde{c}. \end{aligned}$$

For the zero extension $\widetilde{v_\infty}$ of v_∞ from B_3^+ to B_3, we note that

$$\widetilde{v_\infty} = v_0 \ \text{ and then } \ D\widetilde{v_\infty} = Dv_0 \ \text{ a.e. in } B_3, \tag{4.264}$$

since $v_k \to v_\infty$ a.e. in B_3^+ and $\widetilde{v_k} \to v_0$ a.e. in B_3.

We now choose a cut-off function $\zeta \in C_0^\infty(B_2)$ such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in B_1 and $|D\zeta| \le 2$. Then we find that

$$\begin{aligned} &\int_{B_1} \langle A_0(D\widetilde{v_k}) - A_0(Dv_0), D\widetilde{v_k} - Dv_0\rangle\,dx \\ &\quad\le \int_{B_2} \zeta\langle A_0(D\widetilde{v_k}) - A_0(Dv_0), D\widetilde{v_k} - Dv_0\rangle\,dx \\ &\quad= \int_{B_2} \zeta\langle A_0(D\widetilde{v_k}), D\widetilde{v_k} - Dv_0\rangle\,dx + \int_{B_2} \zeta\langle A_0(Dv_0), D\widetilde{v_k} - Dv_0\rangle\,dx \\ &\quad=: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{4.265}$$

We first estimate I_1 as follows:

$$\begin{aligned} \mathrm{I}_1 &= \int_{B_2} \zeta\langle A_0(D\widetilde{v_k}), D\widetilde{v_k} - Dv_0\rangle\,dx \\ &= \int_{B_2} \langle A_0(D\widetilde{v_k}), D(\zeta(\widetilde{v_k} - v_0))\rangle\,dx - \int_{B_2} \langle A_0(D\widetilde{v_k}), D\zeta\rangle(\widetilde{v_k} - v_0)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega_2^k} \langle A_0(Dv_k), D(\zeta(\widetilde{v_k} - v_0))\rangle \, dx - \int_{B_2} \langle A_0(D\widetilde{v_k}), D\zeta\rangle(\widetilde{v_k} - v_0) \, dx \\
&= -\int_{B_2} \langle A_0(D\widetilde{v_k}), D\zeta\rangle(\widetilde{v_k} - v_0) \, dx,
\end{aligned}
$$

since one can take $\zeta(\widetilde{v_k} - v_0) \in W_0^{1,p}(\Omega_2^k)$ as a test function in the problem (4.259). From (4.194), (2.1) and (2.3), we see that

$$
\begin{aligned}
H_0^*\left(|A_0(D\widetilde{v_k})|\right) &\le H_0^*\left(L\frac{H_0(|D\widetilde{v_k}|)}{|D\widetilde{v_k}|}\right) \\
&\le c\,(L^{\kappa_1^*} + L^{\kappa_2^*})\, H_0^*\left(\frac{H_0(|D\widetilde{v_k}|)}{|D\widetilde{v_k}|}\right) \\
&\le c\,(L^{\kappa_1^*} + L^{\kappa_2^*})\, H_0(D\widetilde{v_k}), \qquad (4.266)
\end{aligned}
$$

for some positive constants κ_1^* and κ_2^* depending only on p and a_0. Then it follows from the Luxemburg norm property, (4.266) and (4.260) that

$$
\begin{aligned}
\|A_0(D\widetilde{v_k})\|_{L^{H_0^*}(B_2)} &\le 1 + \fint_{B_2} H_0^*\left(|A_0(D\widetilde{v_k})|\right) dx \\
&\le 1 + c\fint_{B_3} H_0^*\left(|A_0(D\widetilde{v_k})|\right) dx \\
&\le 1 + c\fint_{B_3} H_0(D\widetilde{v_k}) \, dx \le 1 + c\widetilde{c}, \qquad (4.267)
\end{aligned}
$$

where c is a constant depending on n, p, L and a_0. Hence we obtain

$$
\begin{aligned}
|\mathrm{I}_1| &= \left|\int_{B_2} \langle A_0(D\widetilde{v_k}), D\zeta\rangle(\widetilde{v_k} - v_0) \, dx\right| \\
&\le 2\int_{B_2} |A_0(D\widetilde{v_k})||\widetilde{v_k} - v_0| \, dx \\
&\le 4|B_2|\, \|A_0(D\widetilde{v_k})\|_{L^{H_0^*}(B_2)} \, \|\widetilde{v_k} - v_0\|_{L^{H_0}(B_2)} \\
&\le 4|B_2|(1 + c\widetilde{c})\, \|\widetilde{v_k} - v_0\|_{L^{H_0}(B_2)} \longrightarrow 0 \quad \text{as} \quad k \to \infty, \qquad (4.268)
\end{aligned}
$$

by (2.6), (4.262) and (4.267).

On the other hand, we deduce from (4.262) that

$$\mathrm{I}_2 = \int_{B_2} \langle \zeta A_0(Dv_0), D\widetilde{v}_k - Dv_0 \rangle \, dx \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{4.269}$$

Combining (4.265) with (4.268) and (4.269) yields

$$\lim_{k\to\infty} \int_{B_1} \langle A_0(D\widetilde{v}_k) - A_0(Dv_0), D\widetilde{v}_k - Dv_0 \rangle \, dx = 0.$$

We note from (4.163) with $a(\cdot) = a_0 = 0$ that

$$\langle A_0(\xi) - A_0(\eta), \xi - \eta \rangle \geq \widetilde{\nu} \left[|V_p(\xi) - V_p(\eta)|^2 + a_0 |V_{\log}(\xi) - V_{\log}(\eta)|^2 \right]$$

for every $\xi, \eta \in \mathbb{R}^n$, where $\widetilde{\nu}$ is a positive constant depending on n, p and ν. Therefore, we conclude that

$$\lim_{k\to\infty} \int_{B_1} \left(|V_p(Dv_k) - V_p(Dv_0)|^2 + a_0 |V_{\log}(Dv_k) - V_{\log}(Dv_0)|^2 \right) dx = 0,$$

which is contrary to (4.261). This finishes the proof. $\square$

The following lemma is a direct consequence of Lemma 4.2.16 and Lemma 4.2.20.

Lemma 4.2.21. *Let $u \in W_0^{1,1}(\Omega)$ be the distributional solution to (4.160) with (4.176). Then for any $\varepsilon > 0$, there exists a positive constant $\delta = \delta(n, p, L, \|a\|_{L^\infty(\Omega)}, \varepsilon) \in (0, \frac{1}{312})$ such that if (4.216) holds and if $h_i \in W^{1,1}(\Omega_i^5)$ is the distributional solution to (4.217) with $H(x, Dh_i) \in L^1(\Omega_i^5)$, $w_i \in W^{1,1}(\Omega_i^4)$ is the distributional solution to (4.227) with $H(x_{i,M}, Dw_i) \in L^1(\Omega_i^4)$ and $v_i \in W^{1,1}(\Omega_i^3)$ is the distributional solution to (4.241) with $H(x_{i,M}, Dv_i) \in L^1(\Omega_i^3)$, then there exists a distributional solution $\overline{v}_i \in W^{1,1}(B_i^{2+})$ to*

$$\begin{cases} \operatorname{div} \widetilde{A}_0(D\overline{v}_i) &= 0 \quad \text{in} \quad B_i^{2+}, \\ \overline{v}_i &= 0 \quad \text{on} \quad T_i^2 \end{cases} \tag{4.270}$$

such that

$$\fint_{\Omega_i^1} \left(|V_p(Dv_i) - V_p(D\overline{v}_i)|^2 + a(x_{i,M}) |V_{\log}(Dv_i) - V_{\log}(D\overline{v}_i)|^2 \right) dx \leq \varepsilon\lambda \tag{4.271}$$

and

$$\sup_{\Omega_i^1} H(x_{i,M}, D\overline{v}_i) \leq c\lambda \tag{4.272}$$

for some positive constant $c = c(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$, *where* $\overline{v}_i$ *is extended by zero from* B_i^{2+} *to* $B_i^2 \supset \Omega_i^2$.

Remark 4.2.22. *For the interior case, Lemma 4.2.17 and Lemma 4.218 with* Ω_i^j *replaced by* B_i^{j-1}*, respectively, still hold. Furthermore, we have the interior Lipschitz regularity for the solution* v_i *to the reference problem (4.227). In fact, it follows from Lemma 4.2.15 and (4.242) that*

$$\sup_{B_i^1} H(x_{i,M}, Dv_i) \leq c\lambda, \tag{4.273}$$

where $x_{i,M} \in \overline{B_i^3}$ *is a point such that*

$$a(x_{i,M}) = \sup_{x \in B_i^3} a(x) \tag{4.274}$$

and $c = c(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ *is a positive constant.*

4.2.5 Proof of Theorem 4.2.4

In this subsection we establish a global gradient estimate for the distributional solution to (4.160) with $H(x, Du) \in L^1(\Omega)$. We use the comparison results which we have shown in the previous subsection and apply the covering arguments which we have discussed in Subsection 4.2.3 to prove the main result as follows.

Proof of Theorem 4.2.4. We first choose a universal constant

$$\widetilde{R} = \widetilde{R}(n, p, \nu, L, \omega(\cdot), \|H(\cdot, F)\|_{L^1(\Omega)}, R_0, \varepsilon) > 0 \tag{4.275}$$

so that the conditions (4.197), (4.225) and (4.239) hold true. According to the covering argument presented in Subsection 4.2.3, there exists a countable family of disjoint sets $\{\Omega_{\rho_i}(y_i)\}_{i=1}^{\infty}$ satisfying (4.204) and (4.205). For the interior case, we know from (4.175), Lemma 4.2.17, Lemma 4.2.18, Lemma 4.2.19 and Remark 4.2.22 that for any $0 < \varepsilon < 1$, there exists a small constant $\delta = \delta(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$ so that one can find $h_i \in W^{1,1}(B_i^4)$,

$w_i \in W^{1,1}(B_i^3)$ and $v_i \in W^{1,1}(B_i^2)$ satisfying

$$\begin{aligned}
&\fint_{B_i^4} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \fint_{B_i^3} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\
&\qquad + \fint_{B_i^2} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \\
&\quad \leq \varepsilon\lambda
\end{aligned} \tag{4.276}$$

and

$$\sup_{B_i^1} \left(|V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dv_i)|^2\right) \leq c_1\lambda, \tag{4.277}$$

where $c_1 = c_1(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ is a positive constant and $x_{i,M} \in \overline{B_i^3}$ is a point such that $a(x_{i,M}) = \sup_{x \in B_i^3} a(x)$.

On the other hand, for the boundary case, we see from (4.175), Lemma 4.2.17, Lemma 4.2.18, Lemma 4.2.19, Lemma 4.2.20 and Lemma 4.2.21 that for any $0 < \varepsilon < 1$, there exists $\delta = \delta(n, p, \nu, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$ so that one can find $h_i \in W^{1,1}(\Omega_i^5)$, $w_i \in W^{1,1}(\Omega_i^4)$, $v_i \in W^{1,1}(\Omega_i^3)$ and $\overline{v}_i \in W^{1,1}(\Omega_i^2)$ satisfying

$$\begin{aligned}
&\fint_{\Omega_i^5} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \fint_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\
&\qquad + \fint_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \\
&\qquad + \fint_{\Omega_i^1} \left(|V_p(Dv_i) - V_p(D\overline{v}_i)|^2 + a(x_{i,M})|V_{\log}(Dv_i) - V_{\log}(D\overline{v}_i)|^2\right) dx \\
&\quad \leq \varepsilon\lambda
\end{aligned} \tag{4.278}$$

and

$$\sup_{\Omega_i^1} \left(|V_p(D\overline{v}_i)|^2 + a(x_{i,M})|V_{\log}(D\overline{v}_i)|^2\right) \leq c_2\lambda, \tag{4.279}$$

where $c_2 = c_2(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ is a positive constant and $x_{i,M} \in \overline{\Omega_i^4}$ is a point such that $a(x_{i,M}) = \sup\limits_{x \in \Omega_i^4} a(x)$.

Consequently, for any $0 < \varepsilon < 1$, there exists a small constant

$$\delta = \delta(n, p, \nu, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0 \tag{4.280}$$

such that (4.276)-(4.279) hold true.

Let us now estimate $|\Omega_{\rho_i}(y_i)|$. It follows from (4.205) that

$$\begin{aligned}
\lambda|\Omega_{\rho_i}(y_i)| &= \int_{\Omega_{\rho_i}(y_i)} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\
&\le \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, Du(x)) > \frac{\lambda}{4}\}} H(x, Du(x))\, dx + \frac{\lambda}{4} |\Omega_{\rho_i}(y_i)| \\
&\quad + \frac{1}{\delta} \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, F(x)) > \frac{\delta\lambda}{4}\}} H(x, F(x))\, dx + \frac{\lambda}{4} |\Omega_{\rho_i}(y_i)|,
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\lambda}{2} |\Omega_{\rho_i}(y_i)| \le \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, Du) > \frac{\lambda}{4}\}} & H(x, Du)\, dx \\
&+ \frac{1}{\delta} \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, F) > \frac{\delta\lambda}{4}\}} H(x, F)\, dx.
\end{aligned} \tag{4.281}$$

For the interior case, using (4.175), (4.274) and (4.277), we have

$$\begin{aligned}
&8c_1\lambda |\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}| + \int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}} H(x, Du)\, dx \\
&\le 2 \int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}} H(x, Du)\, dx \\
&\le 2 \int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}} \left(|V_p(Du)|^2 + a(x) |V_{\log}(Du)|^2 \right) dx \\
&\le 8 \Bigg[\int_{B_{5\rho_i}(y_i)} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \\
&\qquad + \int_{B_{5\rho_i}(y_i)} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \int_{B_{5\rho_i}(y_i)} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \\
&\qquad + \int_{\{x \in B_{5\rho_i}(y_i) : H(x,Du) > 8c_1\lambda\}} \left(|V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dv_i)|^2\right) dx \Bigg] \\
&\le 8 \Bigg[\int_{B_i^4} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \int_{B_i^3} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\
&\qquad + \int_{B_i^2} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \Bigg] \\
&\quad + 8c_1\lambda |\{x \in B_{5\rho_i}(y_i) : H(x,Du) > 8c_1\lambda\}|,
\end{aligned} \tag{4.282}
$$

where we have used the following elementary inequality:

$$
(t_1 + t_2 + \cdots + t_N)^2 \le N(t_1^2 + t_2^2 + \cdots + t_N^2), \tag{4.283}
$$

for any $N \in \mathbb{N}$ and $t_1, t_2, \cdots, t_N \in \mathbb{R}$. Then it follows from (4.282) and (4.276) that

$$
\begin{aligned}
&\int_{\{x \in B_{5\rho_i}(y_i) : H(x,Du) > 8c_1\lambda\}} H(x,Du)\, dx \\
&\le 8 \Bigg[\int_{B_i^4} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \int_{B_i^3} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\
&\qquad + \int_{B_i^2} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \Bigg] \\
&\le 8|B_i^4| \Bigg[\fint_{B_i^4} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \fint_{B_i^3} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \fint_{B_i^2} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx\Bigg] \\
&\le 8|B_i^4|\varepsilon\lambda \\
&= 8 \cdot 20^n |B_i^0|\varepsilon\lambda. \qquad (4.284)
\end{aligned}
$$

Now for the boundary case, by a similar argument, we deduce from (4.175), (4.221), (4.279) and (4.283) that

$$
\begin{aligned}
&10c_2\lambda|\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}| \\
&\qquad + \int_{\{x\in\Omega_{5\rho_i}(y_i):H(x,Du)>10c_2\lambda\}} H(x, Du)\, dx \\
&\quad \le 2 \int_{\{x\in\Omega_{5\rho_i}(y_i):H(x,Du)>10c_2\lambda\}} H(x, Du)\, dx \\
&\quad \le 2 \int_{\{x\in\Omega_{5\rho_i}(y_i):H(x,Du)>10c_2\lambda\}} \left(|V_p(Du)|^2 + a(x)|V_{\log}(Du)|^2\right) dx \\
&\quad \le 10 \Bigg[\int_{\Omega_i^5} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \int_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\
&\qquad + \int_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \\
&\qquad + \int_{\Omega_i^1} \left(|V_p(Dv_i) - V_p(D\overline{v}_i)|^2 + a(x_{i,M})|V_{\log}(Dv_i) - V_{\log}(D\overline{v}_i)|^2\right) dx\Bigg] \\
&\qquad + 10c_2\lambda|\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}|. \qquad (4.285)
\end{aligned}
$$

Then we obtain from (4.285) and (4.278) that

$$
\begin{aligned}
&\int_{\{x\in\Omega_{5\rho_i}(y_i):H(x,Du)>10c_2\lambda\}} H(x, Du)\, dx \\
&\le 10 \Bigg[\int_{\Omega_i^5} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \int_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \int_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \\
&\qquad + \int_{\Omega_i^1} \left(|V_p(Dv_i) - V_p(D\overline{v}_i)|^2 + a(x_{i,M})|V_{\log}(Dv_i) - V_{\log}(D\overline{v}_i)|^2\right) dx\Bigg] \\
&\le 10|\Omega_i^5| \Bigg[\fint_{\Omega_i^5} \left(|V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2\right) dx \\
&\qquad + \fint_{\Omega_i^4} \left(|V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2\right) dx \\
&\qquad + \fint_{\Omega_i^3} \left(|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2\right) dx \\
&\qquad + \fint_{\Omega_i^1} \left(|V_p(Dv_i) - V_p(D\overline{v}_i)|^2 + a(x_{i,M})|V_{\log}(Dv_i) - V_{\log}(D\overline{v}_i)|^2\right) dx\Bigg] \\
&\le 10|\Omega_i^5|\varepsilon\lambda \\
&\le 10 \cdot 300^n |\Omega_i^0|\varepsilon\lambda, \qquad (4.286)
\end{aligned}
$$

where for the last inequality we have used the fact that

$$
|\Omega_i^5| \le |B_{130\rho_i}(y_i)| = 130^n |B_{\rho_i}(y_i)| \le 130^n \left(\frac{16}{7}\right)^n |\Omega_{\rho_i}(y_i)| \le 300^n |\Omega_i^0|,
$$

which follows from (4.214) and the measure density condition (4.171).

Consequently, in either case, we deduce from (4.284) and (4.286) that

$$
\int_{\{x\in\Omega_{5\rho_i}(y_i):H(x,Du)>c_0\lambda\}} H(x,Du)\,dx \le 10\cdot 300^n \varepsilon\lambda|\Omega_{\rho_i}(y_i)|, \qquad (4.287)
$$

where $c_0 = \max\{8c_1, 10c_2\}$ is a universal constant. Combining (4.281) and (4.287) gives

$$
\begin{aligned}
&\int_{\{x\in\Omega_{5\rho_i}(y_i):H(x,Du)>c_0\lambda\}} H(x,Du)\,dx \\
&\qquad \le 20\cdot 300^n \varepsilon \Bigg[\int_{\{x\in\Omega_{\rho_i}(y_i):H(x,Du)>\frac{\lambda}{4}\}} H(x,Du)\,dx
\end{aligned}
$$

$$+\frac{1}{\delta}\int_{\{x\in\Omega_{\rho_i}(y_i):H(x,F)>\frac{\delta\lambda}{4}\}} H(x,F)\,dx\Bigg]. \quad (4.288)$$

For the sake of simplicity as in (4.198), let us denote the upper level set of $H(x,F)$ by

$$\Xi(\lambda,s) := \{x\in\Omega_s : H(x,F(x))>\lambda\}, \quad \frac{R}{2}\le s\le R, \quad \lambda>0. \quad (4.289)$$

Since the family $\{\Omega_{\rho_i}(y_i)\}_{i=1}^{\infty}$ is disjoint, it follows from (4.204), (4.207) and (4.288) that

$$\begin{aligned}
&\int_{E(c_0\lambda,r_1)} H(x,Du)\,dx \\
&\quad\le \sum_{i\ge1}\int_{\{x\in\Omega_{5\rho_i}(y_i):H(x,Du)>c_0\lambda\}} H(x,Du)\,dx \\
&\quad\le 300^{n+1}\varepsilon\Bigg[\sum_{i\ge1}\int_{\{x\in\Omega_{\rho_i}(y_i):H(x,Du)>\frac{\lambda}{4}\}} H(x,Du)\,dx \\
&\qquad\qquad +\frac{1}{\delta}\sum_{i\ge1}\int_{\{x\in\Omega_{\rho_i}(y_i):H(x,F)>\frac{\delta\lambda}{4}\}} H(x,F)\,dx\Bigg] \\
&\quad= 300^{n+1}\varepsilon\Bigg[\int_{\bigcup_{i\ge1}\{x\in\Omega_{\rho_i}(y_i):H(x,Du)>\frac{\lambda}{4}\}} H(x,Du)\,dx \\
&\qquad\qquad +\frac{1}{\delta}\int_{\bigcup_{i\ge1}\{x\in\Omega_{\rho_i}(y_i):H(x,F)>\frac{\delta\lambda}{4}\}} H(x,F)\,dx\Bigg] \\
&\quad\le 300^{n+1}\varepsilon\Bigg[\int_{E\left(\frac{\lambda}{4},r_2\right)} H(x,Du)\,dx+\frac{1}{\delta}\int_{\Xi\left(\frac{\delta\lambda}{4},r_2\right)} H(x,F)\,dx\Bigg].
\end{aligned}$$

After a change of variable with respect to λ, we conclude that

$$\begin{aligned}
&\int_{E(\lambda,r_1)} H(x,Du)\,dx \\
&\quad\le 300^{n+1}\varepsilon\Bigg[\int_{E\left(\frac{\lambda}{4c_0},r_2\right)} H(x,Du)\,dx+\frac{1}{\delta}\int_{\Xi\left(\frac{\delta\lambda}{4c_0},r_2\right)} H(x,F)\,dx\Bigg], \quad (4.290)
\end{aligned}$$

whenever $\lambda > c_0\lambda_0$, where λ_0 has been defined in (4.201).

To estimate $H(x,Du)^\gamma$, we recall that Fubini's theorem yields

$$(\gamma-1)\int_0^M \lambda^{\gamma-2}\int_{E(\lambda,r_1)} H(x,Du)\,dx\,d\lambda = \int_{\Omega_{r_1}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx$$

for any $M > 0$, where $H(x,Du)_M := \min\{H(x,Du),M\}$ is the truncated function of $H(x,Du)$. Here, we note that the right-hand side of the above identity is finite, as $H(x,Du) \in L^1(\Omega)$ and the truncated function $H(x,Du)_M$ is bounded. Then, for $M > c_0\lambda_0$, it follows from (4.290) that

$$\begin{aligned}
&\int_{\Omega_{r_1}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\
&\quad = (\gamma-1)\int_0^{c_0\lambda_0} \lambda^{\gamma-2}\int_{E(\lambda,r_1)} H(x,Du)\,dx\,d\lambda \\
&\qquad + (\gamma-1)\int_{c_0\lambda_0}^M \lambda^{\gamma-2}\int_{E(\lambda,r_1)} H(x,Du)\,dx\,d\lambda \\
&\quad \le (\gamma-1)\int_0^{c_0\lambda_0} \lambda^{\gamma-2}d\lambda\int_{\Omega_R} H(x,Du)\,dx \\
&\qquad + 300^{n+1}\varepsilon\left[(\gamma-1)\int_{c_0\lambda_0}^M \lambda^{\gamma-2}\int_{E\left(\frac{\lambda}{4c_0},r_2\right)} H(x,Du)\,dx\,d\lambda\right. \\
&\qquad\qquad \left.+\frac{1}{\delta}\cdot(\gamma-1)\int_{c_0\lambda_0}^M \lambda^{\gamma-2}\int_{\Xi\left(\frac{\delta\lambda}{4c_0},r_2\right)} H(x,F)\,dx\,d\lambda\right] \\
&\quad \le (c_0\lambda_0)^{\gamma-1}\int_{\Omega_R} H(x,Du)\,dx \\
&\qquad + 300^{n+1}\varepsilon\left[(\gamma-1)\int_0^M \lambda^{\gamma-2}\int_{E\left(\frac{\lambda}{4c_0},r_2\right)} H(x,Du)\,dx\,d\lambda\right. \\
&\qquad\qquad \left.+\frac{1}{\delta}\cdot(\gamma-1)\int_0^\infty \lambda^{\gamma-2}\int_{\Xi\left(\frac{\delta\lambda}{4c_0},r_2\right)} H(x,F)\,dx\,d\lambda\right].
\end{aligned} \tag{4.291}$$

Utilizing a change of variable and Fubini's theorem, we can calculate the last double integrals in the above display as follows:

$$
\begin{aligned}
&(\gamma-1)\int_0^M \lambda^{\gamma-2}\int_{E\left(\frac{\lambda}{4c_0},r_2\right)} H(x,Du)\,dx\,d\lambda \\
&\quad= (4c_0)^{\gamma-1}(\gamma-1)\int_0^{\frac{M}{4c_0}} \lambda^{\gamma-2}\int_{E(\lambda,r_2)} H(x,Du)\,dx\,d\lambda \\
&\quad\le (4c_0)^{\gamma-1}(\gamma-1)\int_0^{M} \lambda^{\gamma-2}\int_{E(\lambda,r_2)} H(x,Du)\,dx\,d\lambda \\
&\quad= (4c_0)^{\gamma-1}\int_{\Omega_{r_2}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx, \qquad (4.292)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\delta}\cdot(\gamma-1)\int_0^\infty \lambda^{\gamma-2}\int_{\Xi\left(\frac{\delta\lambda}{4c_3},r_2\right)} H(x,F)\,dx\,d\lambda \\
&\quad= \frac{1}{\delta}\left(\frac{4c_0}{\delta}\right)^{\gamma-1}(\gamma-1)\int_0^\infty \lambda^{\gamma-2}\int_{\Xi(\lambda,r_2)} H(x,F)\,dx\,d\lambda \\
&\quad= \frac{1}{\delta}\left(\frac{4c_0}{\delta}\right)^{\gamma-1}\int_{\Omega_{r_2}} [H(x,F)]^\gamma\,dx. \qquad (4.293)
\end{aligned}
$$

Combining (4.291) with (4.292) and (4.293) gives

$$
\begin{aligned}
&\int_{\Omega_{r_1}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\
&\qquad\le (c_0\lambda_0)^{\gamma-1}|\Omega_R| \fint_{\Omega_R} H(x,Du)\,dx \\
&\qquad\quad + 300^{n+1}(4c_0)^{\gamma-1}\varepsilon\left[\int_{\Omega_{r_2}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx\right. \\
&\qquad\qquad\left. + \frac{1}{\delta^\gamma}\int_{\Omega_{r_2}} [H(x,F)]^\gamma\,dx\right].
\end{aligned}
$$

We now take $\varepsilon = \varepsilon(n,p,\nu,L,\|H(\cdot,F)\|_{L^1(\Omega)},\gamma) \in (0,1)$ small enough in order to obtain

$$\begin{aligned}\int_{\Omega_{r_1}} & H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\ &\le \frac{1}{2}\int_{\Omega_{r_2}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\ &\qquad + (c_0\lambda_0)^{\gamma-1}|\Omega_R| \fint_{\Omega_R} H(x,Du)\,dx + \frac{1}{\delta^\gamma}\int_{\Omega_R}[H(x,F)]^\gamma\,dx.\end{aligned}$$

Note that once $\varepsilon \equiv \varepsilon(n,p,\nu,L,\|H(\cdot,F)\|_{L^1(\Omega)},\gamma) \in (0,1)$ is chosen, one can find the corresponding constants $\widetilde{R} \equiv \widetilde{R}(n,p,\nu,L,\omega(\cdot),\|H(\cdot,F)\|_{L^1(\Omega)},\gamma,R_0) > 0$ and $\delta \equiv \delta(n,p,\nu,L,\|a\|_{L^\infty(\Omega)},\|H(\cdot,F)\|_{L^1(\Omega)},\gamma) > 0$, see (4.275) and (4.280) respectively. Recalling the definition of λ_0 in (4.201), we see that

$$\begin{aligned}\lambda_0 &= \frac{400^n r_2^n}{(r_2-r_1)^n}\fint_{\Omega_{r_2}}\left[H(x,Du)+\frac{1}{\delta}H(x,F)\right]dx \\ &= \frac{400^n r_2^n}{(r_2-r_1)^n}\frac{|B_{r_2}|}{|\Omega_{r_2}|}\frac{|B_R|}{|B_{r_2}|}\frac{|\Omega_R|}{|B_R|}\fint_{\Omega_R}\left[H(x,Du)+\frac{1}{\delta}H(x,F)\right]dx \\ &\le \frac{400^n r_2^n}{(r_2-r_1)^n}\left(\frac{16}{7}\right)^n\frac{R^n}{r_2^n}\fint_{\Omega_R}\left[H(x,Du)+\frac{1}{\delta}H(x,F)\right]dx \\ &\le \frac{10^{3n}R^n}{(r_2-r_1)^n}\fint_{\Omega_R}\left[H(x,Du)+\frac{1}{\delta}H(x,F)\right]dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{\Omega_{r_1}} & H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\ &\le \frac{1}{2}\int_{\Omega_{r_2}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \\ &\qquad + \frac{(10^{3n}c_0)^{\gamma-1}R^{n(\gamma-1)}|\Omega_R|}{(r_2-r_1)^{n(\gamma-1)}}\left(\fint_{\Omega_R}\left[H(x,Du)+\frac{1}{\delta}H(x,F)\right]dx\right)^\gamma \\ &\qquad + \frac{1}{\delta^\gamma}\int_{\Omega_R}[H(x,F)]^\gamma\,dx.\end{aligned}$$

We now apply Lemma 2.3.1 when

$$\phi(s) = \int_{\Omega_s} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx, \quad \kappa = n(\gamma-1) > 0 \quad \text{and} \quad \vartheta = \frac{1}{2},$$

to discover that

$$\int_{\Omega_{R/2}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \le c|\Omega_R| \left(\fint_{\Omega_R} [H(x,Du) + H(x,F)]\,dx \right)^{\gamma} + c\int_{\Omega_R} [H(x,F)]^{\gamma}\,dx.$$

Hence we have

$$\fint_{\Omega_{R/2}} H(x,Du)[H(x,Du)_M]^{\gamma-1}\,dx \le c\left(\fint_{\Omega_R} H(x,Du)\,dx \right)^{\gamma} + c\fint_{\Omega_R} [H(x,F)]^{\gamma}\,dx,$$

for some positive constant $c = c(n,p,\nu,L,\|a\|_{L^\infty(\Omega)}, \|H(\cdot,F)\|_{L^1(\Omega)}, \gamma)$. Using Fatou's lemma, we obtain a local estimate up to the boundary as follows:

$$\fint_{\Omega_{R/2}} [H(x,Du)]^{\gamma}\,dx \le c\left(\fint_{\Omega_R} H(x,Du)\,dx \right)^{\gamma} + c\fint_{\Omega_R} [H(x,F)]^{\gamma}\,dx, \tag{4.294}$$

which holds for every $0 < R \le \widetilde{R}$.

Since Ω is bounded in $\mathbb{R}^n$, there exists a finite family of balls $\{B_{\widetilde{R}/2}(x_j)\}_{j=1}^N$ with $x_j \in \Omega$ for $j = 1, \cdots, N$ which covers Ω. This clearly forces

$$\int_{\Omega} [H(x,Du)]^{\gamma}\,dx \le \sum_{j=1}^{N} \int_{\Omega_{\widetilde{R}/2}(x_j)} [H(x,Du)]^{\gamma}\,dx. \tag{4.295}$$

Using the local estimate (4.294) with $R = \widetilde{R}$, the standard energy estimate (4.188), Hölder's inequality and the measure density condition (4.171), we deduce that for each $j = 1, \cdots, N$,

$$\int_{\Omega_{\widetilde{R}/2}(x_j)} [H(x,Du)]^{\gamma}\,dx$$

$$
\begin{aligned}
&\leq c|\Omega_{\widetilde{R}}(x_j)|^{1-\gamma}\left(\int_{\Omega_{\widetilde{R}}(x_j)} H(x,Du)\,dx\right)^{\gamma} + c\int_{\Omega_{\widetilde{R}}(x_j)} [H(x,F)]^{\gamma}\,dx \\
&\leq c|\Omega_{\widetilde{R}}(x_j)|^{1-\gamma}\left(\int_{\Omega} H(x,Du)\,dx\right)^{\gamma} + c\int_{\Omega} [H(x,F)]^{\gamma}\,dx \\
&\leq c|\Omega_{\widetilde{R}}(x_j)|^{1-\gamma}\left(\int_{\Omega} H(x,F)\,dx\right)^{\gamma} + c\int_{\Omega} [H(x,F)]^{\gamma}\,dx \\
&\leq c(|\Omega_{\widetilde{R}}(x_j)|^{1-\gamma}|\Omega|^{\gamma-1}+1)\int_{\Omega} [H(x,F)]^{\gamma}\,dx \\
&\leq c(|B_{\widetilde{R}}(x_j)|^{1-\gamma}|\Omega|^{\gamma-1}+1)\int_{\Omega} [H(x,F)]^{\gamma}\,dx. \qquad (4.296)
\end{aligned}
$$

We note that the constant $c \equiv c(n,p,\nu,L,\|a\|_{L^\infty(\Omega)},\|H(\cdot,F)\|_{L^1(\Omega)},\gamma)$ in the above display is independent of j. Then it follows from (4.295) and (4.296) that

$$
\begin{aligned}
\int_{\Omega} [H(x,Du)]^{\gamma}\,dx &\leq \sum_{j=1}^{N} c(|B_{\widetilde{R}}(x_j)|^{1-\gamma}|\Omega|^{\gamma-1}+1)\int_{\Omega} [H(x,F)]^{\gamma}\,dx \\
&= cN(|B_{\widetilde{R}}|^{1-\gamma}|\Omega|^{\gamma-1}+1)\int_{\Omega} [H(x,F)]^{\gamma}\,dx \\
&\leq c\int_{\Omega} [H(x,F)]^{\gamma}\,dx,
\end{aligned}
$$

for a constant $c = c(n,p,\nu,L,\omega(\cdot),\|a\|_{L^\infty(\Omega)},\|H(\cdot,F)\|_{L^1(\Omega)},\gamma,R_0,\Omega) > 0$. This is the desired conclusion (4.178). $\square$

Chapter 5

Regularity results for generalized double phase functionals

In this chapter, we are concerned with the functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{F}(v,\Omega) := \int_\Omega [G(|Dv|) + a(x)H(|Dv|)]\,dx, \tag{5.1}$$

where $G, H : [0,\infty) \to [0,\infty)$ are Young functions satisfying a suitable gap condition, see (5.29), $a : \Omega \to [0,\infty)$ is a continuous function, and Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$.

The main object of this chapter is to investigate an optimal condition on the modulating coefficient $a(\cdot)$ in the functional (5.1) under which the Hölder regularity result holds for local quasi-minimizers. The method used in this chapter is influenced by [8, 38, 39]. For the Hölder continuity of quasi-minimizers, we first derive a Caccioppoli type inequality which is similar to the one that holds for the functional $v \mapsto \displaystyle\int_\Omega G(|Dv|)\,dx$ by using the condition (5.46) on the modulus of continuity of $a(\cdot)$. We then consider a sequence of nested and shrinking balls $\{B_{4^{-i}r_0}\}_{i=0}^{\infty}$ in order to control the oscillation of quasi-minimizers along the sequence of balls. Here we should verify for each ball whether the condition (5.46) holds true. If this condition holds true for every ball, then we obtain the Hölder continuity of quasi-minimizers. Otherwise, we reduce the oscillation until we reach the exit time for ball $B_{4^{-j}r_0}$, and then we use the existing regularity theory, see Lemma 5.3.9, for the frozen functional

$$v \in W^{1,1}(B_{4^{-j}r_0}) \mapsto \int_{B_{4^{-j}r_0}} [G(|Dv|) + a_0 H(|Dv|)]\, dx, \quad a_0 = \sup_{B_{4^{-j}r_0}} a(\cdot).$$

For the proof of the Harnack inequality, we first deduce the weak Harnack inequality and the local sup-estimates under the assumption (5.46). Then we apply the exit time argument as above to obtain the desired inequality.

5.1 Preliminaries

5.1.1 Orlicz spaces and Musielak-Orlicz spaces

A Young function $\Phi : [0,\infty) \to [0,\infty)$ is an increasing convex function satisfying

$$\Phi(0) = 0,\ \lim_{t\to\infty} \Phi(t) = \infty,\ \lim_{t\to 0+} \frac{\Phi(t)}{t} = 0,\ \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

Definition 5.1.1. *Let Φ be a Young function.*

1. *Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a positive number $\Delta_2(\Phi)$ such that $\Phi(2t) \le \Delta_2(\Phi)\,\Phi(t)$ for all $t \ge 0$.*

2. *Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(\nabla_2(\Phi)\,t) \ge 2\nabla_2(\Phi)\,\Phi(t)$ for all $t \ge 0$.*

3. *We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.*

We note that if $\Phi \in \Delta_2$, then $\Delta_2(\Phi) > 2$. Indeed, by the convexity of Φ, we get

$$\Phi(2t) \le \Delta_2(\Phi)\,\Phi(t) \le \frac{\Delta_2(\Phi)}{2}\,\Phi(2t) \ \text{ for all } t \ge 0, \tag{5.2}$$

and hence $\Delta_2(\Phi) \ge 2$. If $\Delta_2(\Phi) = 2$, then it follows from (5.2) that $\Phi(2t) = 2\Phi(t)$ for all $t \ge 0$, and so $\Phi(t) \equiv \Phi(1)\,t$ is not a Young function. Thus $\Delta_2(\Phi) > 2$.

For a given Young function Φ, we define the complementary Young function Φ^* of Φ by

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \geq 0\}.$$

We remark that Φ^* satisfies all the conditions to be a Young function and that $(\Phi^*)^* = \Phi$. Moreover, $\Phi \in \nabla_2$ if and only if $\Phi^* \in \Delta_2$ with $2\nabla_2(\Phi) = \Delta_2(\Phi^*)$.

We will use the following basic properties of Young functions satisfying Δ_2 and ∇_2 conditions, see for instance [5, 101, 106].

Lemma 5.1.2. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$.*

1. *For any $1 \leq \Lambda < \infty$ and $t \geq 0$, we have*

$$\Phi(\Lambda t) \leq \Delta_2(\Phi)\Lambda^{\log_2 \Delta_2(\Phi)}\, \Phi(t). \tag{5.3}$$

2. *For any $0 < \lambda \leq 1$ and $t \geq 0$, we have*

$$\Phi(\lambda t) \leq 2\nabla_2(\Phi)\lambda^{1+\log_{\nabla_2(\Phi)} 2}\, \Phi(t). \tag{5.4}$$

3. *(Young's inequality) For any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$st \leq \varepsilon\Phi(s) + c\Phi^*(t), \quad \forall s, t \geq 0. \tag{5.5}$$

4. *If $\Phi \in C^1([0, \infty))$, then for any $t \geq 0$, we have*

$$c_1^{-1}\Phi(t) \leq t\Phi'(t) \leq c_1\Phi(t) \tag{5.6}$$

and

$$\Phi^*\left(\Phi'(t)\right) \leq c_2\Phi(t) \tag{5.7}$$

for some constants $c_1, c_2 > 1$ depending only on $\Delta_2(\Phi)$ and $\nabla_2(\Phi)$.

5. *(A modified form of Young's inequality) If $\Phi \in C^1([0, \infty))$, then for any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$s\Phi'(t) \leq \varepsilon\Phi(s) + c\Phi(t), \quad \forall s, t \geq 0. \tag{5.8}$$

For a Young function Φ, the Orlicz class $K^\Phi(\Omega; \mathbb{R}^N)$, $N \in \mathbb{N}$, consists of all measurable functions $v : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(|v(x)|)\, dx < +\infty.$$

The Orlicz space $L^\Phi(\Omega; \mathbb{R}^N)$ is the vector space generated by the Orlicz class $K^\Phi(\Omega; \mathbb{R}^N)$. If $\Phi \in \Delta_2$, then $K^\Phi(\Omega; \mathbb{R}^N) = L^\Phi(\Omega; \mathbb{R}^N)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} = \inf \left\{ \sigma > 0 : \int_\Omega \Phi\left(\frac{|v(x)|}{\sigma} \right) dx \leq 1 \right\}.$$

For $N = 1$, we simply write $L^\Phi(\Omega) := L^\Phi(\Omega; \mathbb{R})$.

We state some relevant inequalities regarding the Luxemburg norm, see [106].

Lemma 5.1.3. *Let Φ be a Young function with $\Phi \in \Delta_2$.*

1. $\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} \leq 1 \implies \int_\Omega \Phi(|v|)\, dx \leq \|v\|_{L^\Phi(\Omega;\mathbb{R}^N)}$.
2. $\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} \geq 1 \implies \int_\Omega \Phi(|v|)\, dx \geq \|v\|_{L^\Phi(\Omega;\mathbb{R}^N)}$.
3. $\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} \leq 1 \iff \int_\Omega \Phi(|v|)\, dx \leq 1$.
4. $0 < \|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} < \infty \implies \int_\Omega \Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)}} \right) dx = 1$.
5. *(Hölder's inequality) For any $v \in L^\Phi(\Omega)$ and $w \in L^{\Phi^*}(\Omega)$,*

$$\int_\Omega |vw|\, dx \leq 2\, \|v\|_{L^\Phi(\Omega)}\, \|w\|_{L^{\Phi^*}(\Omega)}\,. \tag{5.9}$$

We now introduce a partial order relation between Young functions, see [121], and present a series of lemmas which will be used frequently throughout the chapter.

Definition 5.1.4. *Let Φ_1, Φ_2 be Young functions. We shall write*

$$\Phi_1 \prec \Phi_2$$

if $\Phi_2 \circ \Phi_1^{-1}$ *is a Young function.*

Lemma 5.1.5. *Let* Φ_1, Φ_2 *be Young functions with* $\Phi_1 \prec \Phi_2$. *Then*

$$\Phi_1(t) \leq \frac{1}{\left(\Phi_2 \circ \Phi_1^{-1}\right)(1)} \Phi_2(t), \quad \forall t \geq \Phi_1^{-1}(1). \tag{5.10}$$

Proof. We first note that for a Young function Φ, there holds

$$\Phi(1)\, s \leq \Phi(s), \quad \forall s \geq 1.$$

Indeed, this follows from the convexity of Φ. Since $\Phi_1 \prec \Phi_2$, we have

$$\left(\Phi_2 \circ \Phi_1^{-1}\right)(1)\, s \leq \left(\Phi_2 \circ \Phi_1^{-1}\right)(s), \quad \forall s \geq 1.$$

Setting $t = \Phi_1^{-1}(s)$, we obtain the desired conclusion (5.10). □

Corollary 5.1.6. *Let* Φ_1, Φ_2 *be Young functions with* $\Phi_1 \prec \Phi_2$. *Then*

$$\Phi_1(t) \leq c\left(\Phi_2(t) + 1\right), \quad \forall t \geq 0, \tag{5.11}$$

where c *is a positive constant depending only on* Φ_1 *and* Φ_2.

Lemma 5.1.7. *Let* Φ_1, Φ_2 *be Young functions with* $\Phi_1 \prec \Phi_2$. *Then the function*

$$t \mapsto \left(\frac{\Phi_2}{\Phi_1}\right)(t) = \frac{\Phi_2(t)}{\Phi_1(t)}$$

is non-decreasing.

Proof. We first note that the function $\dfrac{\Phi_2}{\Phi_1}$ is non-decreasing if and only if the function $\dfrac{\Phi_2}{\Phi_1} \circ \Phi_1^{-1}$ is non-decreasing, as $t \mapsto \Phi_1(t)$ is increasing and continuous. Since $\Phi_1 \prec \Phi_2$, we see that $\Phi_2 \circ \Phi_1^{-1}$ is a Young function. Hence, it follows from the convexity of $\Phi_2 \circ \Phi_1^{-1}$ that the function

$$t \mapsto \left(\frac{\Phi_2}{\Phi_1} \circ \Phi_1^{-1}\right)(t) = \frac{\left(\Phi_2 \circ \Phi_1^{-1}\right)(t)}{t}$$

is non-decreasing. □

Lemma 5.1.8. *Let $\Phi \in C^1([0,\infty)) \cap C^2((0,\infty))$ be a Young function satisfying*

$$\frac{1}{c_\Phi} \leq \frac{t\Phi''(t)}{\Phi'(t)} \leq c_\Phi, \quad \forall t > 0, \tag{5.12}$$

for some $c_\Phi \geq 1$. Then,

1. *$\Phi \in \Delta_2 \cap \nabla_2$, and the constants $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ depend only on c_Φ.*
2. *for any $1 \leq \Lambda < \infty$ and $t \geq 0$, we have*

$$\Phi(\Lambda t) \leq \Lambda^{c_\Phi+1}\Phi(t). \tag{5.13}$$

3. *for any $0 < \lambda \leq 1$ and $t \geq 0$, we have*

$$\Phi(\lambda t) \leq \lambda^{\frac{1}{c_\Phi}+1}\Phi(t). \tag{5.14}$$

Proof. (1) We see from (5.12) that

$$\left(\frac{1}{c_\Phi}+1\right)\Phi'(t) \leq t\Phi''(t) + \Phi'(t) \leq (c_\Phi+1)\Phi'(t), \quad \forall t > 0.$$

Integrating this over the interval $(0,s)$, we get

$$\left(\frac{1}{c_\Phi}+1\right)\Phi(s) \leq s\Phi'(s) \leq (c_\Phi+1)\Phi(s), \quad \forall s > 0. \tag{5.15}$$

Then we obtain

$$\int_t^{2t} \frac{\Phi'(s)}{\Phi(s)}\,ds \leq (c_\Phi+1)\int_t^{2t} \frac{1}{s}\,ds = (c_\Phi+1)\ln 2,$$

which implies that $\Phi(2t) \leq 2^{c_\Phi+1}\Phi(t)$ for all $t \geq 0$.
Next, we shall prove that $\Phi^* \in \Delta_2$. To this end, we observe from the definition of the complementary Young function that

$$\Phi^*(\Phi'(s)) = s\Phi'(s) - \Phi(s), \quad \forall s > 0.$$

Then it follows from (5.15) that

$$\frac{1}{c_\Phi}s\Phi'(s) - \left(\frac{1}{c_\Phi}+1\right)\Phi^*(\Phi'(s)) \le 0 \le c_\Phi s\Phi'(s) - (c_\Phi+1)\Phi^*(\Phi'(s)),$$

and hence

$$\left(\frac{1}{c_\Phi}+1\right)\Phi^*(\Phi'(s)) \le s\Phi'(s) \le (c_\Phi+1)\Phi^*(\Phi'(s)), \quad \forall s>0.$$

Setting $s=(\Phi^*)'(t)$, we deduce that $t=\Phi'(s)$ and

$$\left(\frac{1}{c_\Phi}+1\right)\Phi^*(t) \le t(\Phi^*)'(t) \le (c_\Phi+1)\Phi^*(t), \quad \forall t>0. \tag{5.16}$$

Therefore, we have $\Phi^*(2t)\le 2^{c_\Phi+1}\Phi^*(t)$ for all $t\ge 0$ as above.
Since $\Phi\in\nabla_2$ if and only if $\Phi^*\in\Delta_2$ with $2\nabla_2(\Phi)=\Delta_2(\Phi^*)$, we conclude that $\Phi\in\Delta_2\cap\nabla_2$, and that the constants $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ depend only on c_Φ.

(2) It follows from (5.15) that

$$\int_t^{\Lambda t}\frac{\Phi'(s)}{\Phi(s)}\,ds \le (c_\Phi+1)\int_t^{\Lambda t}\frac{1}{s}\,ds = (c_\Phi+1)\ln\Lambda,$$

which gives (5.13).

(3) From (5.15) we have

$$\int_{\lambda t}^{t}\frac{\Phi'(s)}{\Phi(s)}\,ds \ge \left(\frac{1}{c_\Phi}+1\right)\int_{\lambda t}^{t}\frac{1}{s}\,ds = \left(\frac{1}{c_\Phi}+1\right)\ln\left(\frac{1}{\lambda}\right),$$

and (5.13) follows. □

Lemma 5.1.9. *Let Φ be a Young function with $\Phi\in C^1([0,\infty))\cap C^2((0,\infty))$. If*

$$\frac{t\Phi''(t)}{\Phi'(t)} \le c_\Phi, \quad \forall t>0,$$

for some $c_\Phi\ge 1$, then $t\mapsto\Phi\left(t^{\frac{1}{1+c_\Phi}}\right)$ is a concave function.

Proof. Set $\varphi(t) := \Phi\left(t^{\frac{1}{1+c_\Phi}}\right)$ for $t \geq 0$. Then we have

$$\varphi'(t) = \frac{1}{1+c_\Phi}\Phi'\left(t^{\frac{1}{1+c_\Phi}}\right)t^{-\frac{c_\Phi}{1+c_\Phi}},$$

and hence

$$\begin{aligned}\varphi''(t) &= \frac{1}{(1+c_\Phi)^2}\Phi''\left(t^{\frac{1}{1+c_\Phi}}\right)\left(t^{-\frac{c_\Phi}{1+c_\Phi}}\right)^2 - \frac{c_\Phi}{(1+c_\Phi)^2}\Phi'\left(t^{\frac{1}{1+c_\Phi}}\right)t^{-\frac{c_\Phi}{1+c_\Phi}-1}\\ &= \frac{1}{(1+c_\Phi)^2}\,t^{-\frac{c_\Phi}{1+c_\Phi}-1}\left[t^{\frac{1}{1+c_\Phi}}\,\Phi''\left(t^{\frac{1}{1+c_\Phi}}\right) - c_\Phi\Phi'\left(t^{\frac{1}{1+c_\Phi}}\right)\right] \leq 0\end{aligned}$$

for all $t > 0$. □

We now introduce the Musielak-Orlicz spaces which generalize the Orlicz spaces. Let $\Phi : \Omega \times [0,\infty) \to [0,\infty)$ be a function satisfying the following conditions:

1. $\Phi(x,\cdot)$ is a Young function for every $x \in \Omega$,

2. $\Phi(\cdot,t)$ is a measurable function for every $t \geq 0$.

Such a function $\Phi(x,t)$ is called a Musielak-Orlicz function. As before, we present some definitions and properties regarding Musielak-Orlicz functions.

Definition 5.1.10. *Let Φ be a Musielak-Orlicz function.*

1. *Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a positive number $\Delta_2(\Phi)$ such that $\Phi(x,2t) \leq \Delta_2(\Phi)\,\Phi(x,t)$ for all $x \in \Omega$ and $t \geq 0$.*

2. *Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(x,\nabla_2(\Phi)\,t) \geq 2\nabla_2(\Phi)\,\Phi(x,t)$ for all $x \in \Omega$ and $t \geq 0$.*

3. *We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.*

For a given Musielak-Orlicz function Φ, we define the complementary Φ^* of Φ by for each $x \in \Omega$,

$$\Phi^*(x,t) = \sup\{st - \Phi(x,s) : s \geq 0\}.$$

Then Φ^* satisfies all the conditions to be a Musielak-Orlicz function. Also we note that $(\Phi^*)^* = \Phi$ and that $\Phi \in \nabla_2$ if and only if $\Phi^* \in \Delta_2$ with $2\nabla_2(\Phi) = \Delta_2(\Phi^*)$.

The following lemma can be directly obtained from the definitions of Δ_2-condition, ∇_2-condition and the complementary of Musielak-Orlicz function.

Lemma 5.1.11. *Let Φ be a Musielak-Orlicz function with $\Phi \in \Delta_2 \cap \nabla_2$.*

1. *For any $1 \le \Lambda < \infty$, $t \ge 0$ and $x \in \Omega$, we have*

$$\Phi(x, \Lambda t) \le \Delta_2(\Phi)\Lambda^{\log_2 \Delta_2(\Phi)}\, \Phi(x,t). \tag{5.17}$$

2. *For any $0 < \lambda \le 1$, $t \ge 0$ and $x \in \Omega$, we have*

$$\Phi(x, \lambda t) \le 2\nabla_2(\Phi)\lambda^{1+\log_{\nabla_2(\Phi)} 2}\, \Phi(x,t). \tag{5.18}$$

3. *(Young's inequality) For any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$st \le \varepsilon\Phi(x, s) + c\Phi^*(x, t) \tag{5.19}$$

for all $s, t \ge 0$ and $x \in \Omega$.

For a Musielak-Orlicz function Φ, the Musielak-Orlicz class $K^\Phi(\Omega; \mathbb{R}^N)$, $N \in \mathbb{N}$, consists of all measurable functions $v : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(x, |v(x)|)\, dx < +\infty.$$

The Musielak-Orlicz space $L^\Phi(\Omega; \mathbb{R}^N)$ is the vector space generated by the Musielak-Orlicz class $K^\Phi(\Omega; \mathbb{R}^N)$. If $\Phi \in \Delta_2$, then $K^\Phi(\Omega; \mathbb{R}^N) = L^\Phi(\Omega; \mathbb{R}^N)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} = \inf\left\{\sigma > 0 : \int_\Omega \Phi\left(x, \frac{|v(x)|}{\sigma}\right) dx \le 1\right\}.$$

The Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\Omega; \mathbb{R}^N)$ is the function space of all measurable functions $v \in L^\Phi(\Omega; \mathbb{R}^N)$ such that its distributional gradient vector Dv belongs to $L^\Phi(\Omega; \mathbb{R}^{Nn})$. For $v \in W^{1,\Phi}(\Omega; \mathbb{R}^N)$, we define its norm to be

$$\|v\|_{W^{1,\Phi}(\Omega;\mathbb{R}^N)} = \|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} + \|Dv\|_{L^\Phi(\Omega;\mathbb{R}^{Nn})}.$$

The space $W_0^{1,\Phi}(\Omega;\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\Omega;\mathbb{R}^N)$ in $W^{1,\Phi}(\Omega;\mathbb{R}^N)$. For $N=1$, we simply write $L^\Phi(\Omega) := L^\Phi(\Omega;\mathbb{R})$ and $W^{1,\Phi}(\Omega) := W^{1,\Phi}(\Omega;\mathbb{R})$. For a detailed discussion of the Musielak-Orlicz space and the associated Sobolev space, we refer the reader to [9, 44, 58, 59, 99, 116] and references therein.

5.1.2 Gap conditions

We now consider the double phase functional

$$\mathcal{F}(v,\Omega) = \int_\Omega [G(|Dv|) + a(x)H(|Dv|)]\,dx, \quad v \in W^{1,1}(\Omega),$$

and investigate gap conditions on two Young functions G and H.

In the rest of the chapter we shall use the notation

$$\Psi(x,\xi) := G(|\xi|) + a(x)H(|\xi|), \tag{5.20}$$

when $x \in \Omega$ and $\xi \in \mathbb{R}^n$. By abuse of notation, we will continue to write $\Psi(x,\xi)$ also when $x \in \Omega$ and $\xi \in \mathbb{R}$.

Proposition 5.1.12. *Let $G, H : [0,\infty) \to [0,\infty)$ be Young functions. Suppose that the function $a = a(\cdot) : \Omega \to [0,\infty)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho\to 0+} \omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}} < \infty. \tag{5.21}$$

If $H \succ G^\kappa$ for some $\kappa > 1 + \frac{1}{n}$, then $a(\cdot)$ is a constant function.

Proof. It follows from the condition (5.21) that there exists a constant $L > 0$ such that

$$\omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}} \le L$$

for all $0 < \rho \le 1$. Since $H \succ G^\kappa$, we have

$$\omega(\rho)\frac{(G^\kappa\circ G^{-1})(\rho^{-n})}{\rho^{-n}} \le c\omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}} \le cL \tag{5.22}$$

for all small $\rho > 0$. Here, we see that

$$\omega(\rho)\frac{(G^\kappa \circ G^{-1})(\rho^{-n})}{\rho^{-n}} = \omega(\rho)\frac{[(G \circ G^{-1})(\rho^{-n})]^\kappa}{\rho^{-n}} = \omega(\rho)\rho^{-n(\kappa-1)} \qquad (5.23)$$

Combining (5.22) with (5.23) yields

$$\omega(\rho) \le cL\rho^{n(\kappa-1)}, \quad \forall \rho \le \rho_0, \qquad (5.24)$$

for some small $\rho_0 > 0$. Then we conclude from the definition of the modulus of continuity that

$$\frac{|a(x) - a(y)|}{|x-y|} \le cL|x-y|^{n(\kappa-1)-1} \qquad (5.25)$$

for every $x, y \in \Omega$ with $0 < |x-y| \le \rho_0$. Since $n(\kappa-1)-1 > 0$, it follows immediately that $a(\cdot)$ is a constant function. □

Proposition 5.1.13. *Let $G, H : [0,\infty) \to [0,\infty)$ be Young functions. Suppose that the function $a = a(\cdot) : \Omega \to [0,\infty)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho\to 0+} \omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty. \qquad (5.26)$$

If $H \succ G^\kappa$ for some $\kappa > 2$, then $a(\cdot)$ is a constant function.

Proof. It follows from the condition (5.26) that there exists a constant $L > 0$ such that

$$\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le L$$

for all $0 < \rho \le 1$. We note from the convexity of G that

$$G(1)\, s \le G(s), \quad \forall s \ge 1.$$

Since $H \succ G^\kappa$, we get

$$\begin{aligned}\omega(\rho)\rho^{-(\kappa-1)} &\le c\omega(\rho)\left[G(\rho^{-1})\right]^{\kappa-1} \\ &= c\omega(\rho)\frac{[G(\rho^{-1})]^\kappa}{G(\rho^{-1})} \le c\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le cL\end{aligned}$$

for all small $\rho > 0$. Hence we have

$$\omega(\rho) \leq cL\rho^{\kappa-1}, \quad \forall \rho \leq \rho_0, \tag{5.27}$$

for some small $\rho_0 > 0$. Then it follows from the definition of the modulus of continuity that

$$\frac{|a(x) - a(y)|}{|x - y|} \leq cL|x - y|^{\kappa-2} \tag{5.28}$$

for every $x, y \in \Omega$ with $0 < |x - y| \leq \rho_0$. Since $\kappa > 2$, we conclude that $a(\cdot)$ is a constant function. □

Remark 5.1.14. *If $G(t) \succ t^n$, then it follows from Lemma 5.1.5 and Lemma 5.1.7 that*

$$\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} = \left(\frac{H}{G}\right)\left(G^{-1}(\rho^{-n})\right) \leq \left(\frac{H}{G}\right)(c\rho^{-1}) \leq c\frac{H(\rho^{-1})}{G(\rho^{-1})},$$

and hence the condition (5.26) implies (5.21). On the contrary, if $G(t) \prec t^n$, then

$$\frac{H(\rho^{-1})}{G(\rho^{-1})} = \left(\frac{H}{G}\right)(\rho^{-1}) \leq \left(\frac{H}{G}\right)\left(c\, G^{-1}(\rho^{-n})\right) \leq c\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}},$$

and consequently the condition (5.21) implies (5.26). These agree with the known results in the classical case, see Remark 5.2.2 below.

Throughout the chapter, we assume that $G, H : [0, \infty) \to [0, \infty)$ are Young functions with $G, H \in \Delta_2 \cap \nabla_2$ and

$$G \prec H \prec G^{1+\frac{1}{n}}. \tag{5.29}$$

We remark that $\Psi \in \Delta_2 \cap \nabla_2$. We further assume that $G, H \in C^1([0, \infty)) \cap C^2((0, \infty))$ and there exist constants $c_G, c_H \geq 1$ such that

$$\frac{1}{c_G} \leq \frac{tG''(t)}{G'(t)} \leq c_G \quad \text{and} \quad \frac{1}{c_H} \leq \frac{tH''(t)}{H'(t)} \leq c_H \tag{5.30}$$

hold for all $t > 0$.

5.2 Lavrentiev phenomenon

When considering the functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(x, Dv)\, dx$$

with

$$G(|\xi|) \lesssim F(x,\xi) \lesssim H(|\xi|) + 1, \quad G \prec H,$$

the Lavrentiev phenomenon

$$\inf_{v \in W^{1,G}(\Omega)} \int_\Omega F(x, Dv)\, dx < \inf_{v \in W^{1,H}(\Omega)} \int_\Omega F(x, Dv)\, dx$$

may occur. However, for the functional $\mathcal{F}$ defined in (5.1), there is no Lavrentiev phenomenon under a suitable condition on the modulating coefficient $a(\cdot)$.

Theorem 5.2.1. *Let $\mathcal{F}$ be the functional defined in (5.1).*

1. *If the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying*
$$\limsup_{\rho \to 0+} \omega(\rho) \frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} < \infty, \tag{5.31}$$
then for every function $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and balls $B \Subset \widetilde{B} \Subset \Omega$ with $\mathcal{F}(v, \widetilde{B}) < \infty$, there exists a sequence $\{v_k\} \subset W^{1,\infty}(B)$ such that
$$v_k \longrightarrow v \quad \text{in} \quad W^{1,G}(B) \qquad \text{and} \qquad \mathcal{F}(v_k, B) \longrightarrow \mathcal{F}(v, B). \tag{5.32}$$

2. *If the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying*
$$\limsup_{\rho \to 0+} \omega(\rho) \frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty, \tag{5.33}$$
then for every function $v \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and balls $B \Subset \widetilde{B} \Subset \Omega$ with $\mathcal{F}(v, \widetilde{B}) < \infty$, there exists a sequence $\{v_k\} \subset W^{1,\infty}(B)$ such that
$$v_k \longrightarrow v \quad \text{in} \quad W^{1,G}(B) \qquad \text{and} \qquad \mathcal{F}(v_k, B) \longrightarrow \mathcal{F}(v, B). \tag{5.34}$$

Proof. Let $R > 0$ be the radius of the ball B. Take $\varepsilon_0 \in (0,1)$ in such a way that $B \equiv B_R \Subset B_{R+\varepsilon_0} \Subset \widetilde{B} \Subset \Omega$. Let $\varphi \in C_0^\infty(B_1)$ be a mollifier with $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi\, dx = 1$, and set

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$$

for $x \in B_\varepsilon$ with $\varepsilon > 0$. Then it is obvious that $\varphi_\varepsilon \in C_0^\infty(B_\varepsilon)$, $\int_{\mathbb{R}^n} \varphi_\varepsilon\, dx = 1$, $0 \leq \varphi_\varepsilon \leq c(n)\varepsilon^{-n}$ and $|D\varphi_\varepsilon| \leq c(n)\varepsilon^{-(n+1)}$. Now we define, for $0 < \varepsilon < \varepsilon_0$,

$$v_\varepsilon(x) := (v*\varphi_\varepsilon)(x), \quad a_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} a(y), \quad \Psi_\varepsilon(x,\xi) := G(|\xi|) + a_\varepsilon(x)H(|\xi|)$$

for $x \in B_R$ and $\xi \in \mathbb{R}^n$.

(1) It follows from Jensen's inequality that

$$G\left(|Dv_\varepsilon(x)|\right) = G\left(|Dv * \varphi_\varepsilon(x)|\right) \leq \int_{\mathbb{R}^n} G\left(|Dv(x-y)|\right)\varphi_\varepsilon(y)\, dy \leq c\varepsilon^{-n}$$

for every $x \in B_R$. By the definitions of $a_\varepsilon(\cdot)$, we obtain

$$\begin{aligned}
\Psi(x, Dv_\varepsilon(x)) &\leq |a(x) - a_\varepsilon(x)|\, H\left(|Dv_\varepsilon(x)|\right) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\
&\leq c\omega(\varepsilon) H\left(|Dv_\varepsilon(x)|\right) + \Psi_\varepsilon(x, Dv_\varepsilon(x)).
\end{aligned}$$

We now observe from Lemma 5.1.2 and Lemma 5.1.7 that

$$\begin{aligned}
H\left(|Dv_\varepsilon(x)|\right) &= \left(\frac{H}{G}\right)\left(|Dv_\varepsilon(x)|\right) G\left(|Dv_\varepsilon(x)|\right) \\
&\leq \left(\frac{H}{G}\right)\left(G^{-1}(c\varepsilon^{-n})\right) G\left(|Dv_\varepsilon(x)|\right) \\
&= \frac{(H \circ G^{-1})(c\varepsilon^{-n})}{c\varepsilon^{-n}} G\left(|Dv_\varepsilon(x)|\right) \\
&\leq c\, \frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} G\left(|Dv_\varepsilon(x)|\right) \\
&\leq c\, \frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} \Psi_\varepsilon(x, Dv_\varepsilon(x)).
\end{aligned}$$

Therefore, we see from (5.31) that

$$\Psi(x, Dv_\varepsilon(x)) \leq c\omega(\varepsilon)\frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}}\Psi_\varepsilon(x, Dv_\varepsilon(x)) + \Psi_\varepsilon(x, Dv_\varepsilon(x))$$

$$\leq c\Psi_\varepsilon(x, Dv_\varepsilon(x)). \tag{5.35}$$

By Jensen's inequality, we have

$$\begin{aligned}\Psi_\varepsilon(x, Dv_\varepsilon(x)) &\leq \int_{B_\varepsilon(x)} \Psi_\varepsilon(x, Dv(y))\varphi_\varepsilon(x-y)\, dy \\ &\leq \int_{B_\varepsilon(x)} \Psi(y, Dv(y))\varphi_\varepsilon(x-y)\, dy \\ &= \left[\Psi(\cdot, Dv(\cdot)) * \varphi_\varepsilon\right](x) \\ &=: \left[\Psi(\cdot, Dv(\cdot))\right]_\varepsilon (x). \end{aligned} \tag{5.36}$$

Combining (5.35) and (5.36), we deduce that

$$\Psi(x, Dv_\varepsilon(x)) \leq c\left[\Psi(\cdot, Dv(\cdot))\right]_\varepsilon (x). \tag{5.37}$$

Using the fact that $\left[\Psi(\cdot, Dv(\cdot))\right]_\varepsilon \to \Psi(\cdot, Dv(\cdot))$ strongly in $L^1(B_R)$, we can apply a generalized version of Lebesgue dominated convergence theorem to obtain a sequence of functions $\{v_k\} := \{v_{\varepsilon_k}\} \subset C_0^\infty(B_R)$ satisfying (5.32), for a suitable sequence $\varepsilon_k \to 0$.

(2) Since v is locally bounded in Ω, we have

$$\begin{aligned}|Dv_\varepsilon(x)| = |v * D\varphi_\varepsilon(x)| &\leq \int_{\mathbb{R}^n} |v(x-y)||D\varphi_\varepsilon(y)|\, dy \\ &\leq \|v\|_{L^\infty(\widetilde{B})} \int_{B_\varepsilon} |D\varphi_\varepsilon(y)|\, dy \\ &\leq \|v\|_{L^\infty(\widetilde{B})}\, c(n)\varepsilon^{-(n+1)}|B_\varepsilon| \leq c\varepsilon^{-1}\end{aligned}$$

for every $x \in B_R$. Then we obtain from Lemma 5.1.2 and Lemma 5.1.7 that

$$\begin{aligned}H\left(|Dv_\varepsilon(x)|\right) &= \left(\frac{H}{G}\right)\left(|Dv_\varepsilon(x)|\right) G\left(|Dv_\varepsilon(x)|\right) \\ &\leq \left(\frac{H}{G}\right)\left(c\varepsilon^{-1}\right) G\left(|Dv_\varepsilon(x)|\right) = \frac{H(c\varepsilon^{-1})}{G(c\varepsilon^{-1})} G\left(|Dv_\varepsilon(x)|\right) \\ &\leq c\,\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})} G\left(|Dv_\varepsilon(x)|\right) \leq c\,\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}\Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

As in the proof of (1), it follows from (5.33) and (5.36) that

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\le c\omega(\varepsilon) H\left(|Dv_\varepsilon(x)|\right) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\omega(\varepsilon)\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}\Psi_\varepsilon(x, Dv_\varepsilon(x)) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\left[\Psi(\cdot, Dv(\cdot))\right]_\varepsilon(x).\end{aligned}$$

Again, by a generalized version of Lebesgue dominated convergence theorem, we get a sequence of functions $\{v_k\} := \{v_{\varepsilon_k}\} \subset C_0^\infty(B_R)$ satisfying (5.34), for a suitable sequence $\varepsilon_k \to 0$. □

Remark 5.2.2. *In the special case* $(G(t), H(t)) = (t^p, t^q)$ *with* $1 < p < q$, *and* $a(\cdot) \in C^{0,\alpha}(\Omega)$ *with* $\alpha \in (0, 1]$, *a simple computation shows that*

$$\text{the condition (5.31)} \quad \Longleftrightarrow \quad \frac{q}{p} \le 1 + \frac{\alpha}{n},$$

and

$$\text{the condition (5.33)} \quad \Longleftrightarrow \quad q \le p + \alpha.$$

Therefore Theorem 5.2.1 generalizes [38, Theorem 4.1] and [39, Proposition 3.6]. Moreover, in the case $(G(t), H(t)) = (t^p, t^p \ln(1 + t))$ *with* $p > 1$, *we see that the condition (5.31) and the condition (5.33) are equivalent to*

$$\limsup_{\rho \to 0+} \omega(\rho) \ln\left(\frac{1}{\rho}\right) < \infty.$$

This shows that when $a(\cdot)$ *is log-Hölder continuous, the functional*

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega \left[|Dv|^p + a(x)|Dv|^p \ln(1 + |Dv|)\right] dx, \quad p > 1,$$

has no Lavrentiev phenomenon.

Remark 5.2.3. *The conditions in Theorem 5.2.1 are sharp for the absence of the Lavrentiev phenomenon. Indeed, for any ball* $B \subset \Omega$, *there exist Young functions* G, H *satisfying (5.29), a non-negative coefficient* $a(\cdot)$ *which has a* *modulus of continuity ω satisfying*

$$\lim_{\rho\to 0+} \omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}} = \infty \quad \text{and} \quad \lim_{\rho\to 0+} \omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} = \infty, \tag{5.38}$$

and a boundary datum $v_0 \in W^{1,G}(B)\cap L^\infty(B)$ such that

$$\inf_{v\in v_0+W_0^{1,G}(B)} \mathcal{F}(v,B) < \inf_{v\in v_0+W_0^{1,G}(B)\cap W^{1,H}_{\mathrm{loc}}(B)} \mathcal{F}(v,B). \tag{5.39}$$

That is, local minimizers of $\mathcal{F}$ do not belong to $W^{1,H}_{\mathrm{loc}}(B)$ in general. Moreover, they can be discontinuous.

To see this, let us consider the classical case $G(t)=t^p$, $H(t)=t^q$ and $a(\cdot)\in C^{0,\alpha}(\Omega)$ with $1<p<q$, $\alpha\in(0,1]$ satisfying

$$1<p<n<n+\alpha<q. \tag{5.40}$$

Then it follows from [38, Theorem 4.1] and [55, Section 3] that there exists a coefficient function $a(\cdot)\in C^{0,\alpha}(\Omega)$ and a boundary datum $v_0\in W^{1,p}(B)\cap L^\infty(B)$ such that the Lavrentiev phenomenon (5.39) occurs. Furthermore, we deduce from Remark 5.2.2 and (5.40) that the coefficient function $a(\cdot)$ has a modulus of continuity ω satisfying (5.38).

5.3 Local boundedness and Hölder continuity

In the following, we deal with local quasi-minimizers of $\mathcal{F}$.

Definition 5.3.1. *We say that $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local quasi-minimizer of $\mathcal{F}$ for $Q\geq 1$, or a local Q-minimizer of $\mathcal{F}$, if for any $v\in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $K:=\operatorname{supp}(u-v)\Subset\Omega$, we have $\mathcal{F}(u,K)<+\infty$ and*

$$\mathcal{F}(u,K)\leq Q\mathcal{F}(v,K).$$

If $Q=1$, we say that u is a local minimizer of $\mathcal{F}$.

We remark that if $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional

$$v\in W^{1,1}(\Omega)\mapsto \int_\Omega F(x,v,Dv)\,dx$$

under the assumption that

$$c_1\Psi(x,\xi) \le F(x,z,\xi) \le c_2\Psi(x,\xi)$$

for all $x \in \Omega$, $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ with some constants $0 < c_1 \le 1 \le c_2$, then u is also a local quasi-minimizer of the functional (5.1) with $Q = c_2/c_1 \ge 1$.

To prove the local boundedness of quasi-minimizers of $\mathcal{F}$, we derive the following growth condition on the energy density $\Psi(x,\xi)$ of $\mathcal{F}$.

Lemma 5.3.2. *Suppose that the gap condition (5.29) holds. If $a \in L^\infty(\Omega)$, then*

$$G(|\xi|) \le \Psi(x,\xi) \le c\left(1 + [G(|\xi|)]^{1+\frac{1}{n}}\right), \tag{5.41}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where c is a positive constant depending only on n, G, H and $\|a\|_{L^\infty(\Omega)}$.

Proof. Since $a(\cdot) \ge 0$, it is clear that

$$G(|\xi|) \le G(|\xi|) + a(x)H(|\xi|) = \Psi(x,\xi)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, it follows from Corollary 5.1.6 and (5.29) that

$$\begin{aligned}
\Psi(x,\xi) &= G(|\xi|) + a(x)H(|\xi|) \le G(|\xi|) + \|a\|_{L^\infty(\Omega)} H(|\xi|) \\
&\le \left([G(|\xi|)]^{1+\frac{1}{n}} + 1\right) + c\,\|a\|_{L^\infty(\Omega)}\left([G(|\xi|)]^{1+\frac{1}{n}} + 1\right) \\
&\le c\left([G(|\xi|)]^{1+\frac{1}{n}} + 1\right)
\end{aligned}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. $\square$

We notice that

$$1 + \frac{1}{n} < 1 + \frac{1}{n-1} = 1^*,$$

where 1^* is the Sobolev exponent of 1. The local boundedness of quasi-minimizers of $\mathcal{F}$ now follows from the result of [42, Theorem 2.1].

Theorem 5.3.3 (Local boundedness). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local quasi-minimizer of $\mathcal{F}$ under the assumption (5.29), with $a \in L^\infty_{\mathrm{loc}}(\Omega)$. Then u is locally bounded in Ω.*

We now start the proof of the Hölder continuity of local quasi-minimizers of $\mathcal{F}$. We first define the upper and lower level sets. For $k \in \mathbb{R}$, $\rho > 0$ and a quasi-minimizer u of the functional $\mathcal{F}$, we set

$$A(k,\rho) := \{x \in B_\rho : u(x) > k\} \quad \text{and} \quad A^-(k,\rho) := \{x \in B_\rho : u(x) \le k\}.$$

We state and prove the following Caccioppoli-type inequality.

Lemma 5.3.4 (Caccioppoli inequality)**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$. Then there exists a constant $c = c(Q, \Delta_2(G), \Delta_2(H)) > 0$ such that for any concentric balls $B_{\rho'} \subset B_\rho \subset \Omega$ with $0 < \rho' < \rho < \infty$, and $k \in \mathbb{R}$, we have*

$$\int_{B_{\rho'}} \Psi\left(x, D(u-k)_\pm\right) dx \le c \int_{B_\rho} \Psi\left(x, \frac{(u-k)_\pm}{\rho - \rho'}\right) dx. \tag{5.42}$$

Proof. We note that it suffices to prove the version with $(u-k)_+$, as $-u$ is also a Q-minimizer of $\mathcal{F}$. Let $\eta \in C_0^\infty(B_\rho)$ be a cut-off function with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\rho'}$, and $|D\eta| \le \frac{2}{\rho-\rho'}$. We set $v := u - \eta(u-k)_+$, to be used as a competitor. Note that $\operatorname{supp}(u-v) \subset A(k,\rho)$. Then the Q-minimality of u gives

$$\begin{aligned}
\int_{A(k,\rho')} \Psi(x, Du)\, dx &\le Q \int_{A(k,\rho)} \Psi(x, Dv)\, dx \\
&= Q \int_{A(k,\rho)} \Psi\left(x, (1-\eta)Du - (u-k)_+ D\eta\right) dx \\
&\le c_* \left(\int_{A(k,\rho)\setminus A(k,\rho')} \Psi(x, Du)\, dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx \right)
\end{aligned}$$

for some constant $c_* = c_*(Q, \Delta_2(\Psi)) = c_*(Q, \Delta_2(G), \Delta_2(H)) \ge 1$. We now use the "hole filling" method, that is, we add to both sides the quantity

$$c_* \int_{A(k,\rho')} \Psi(x, Du)\, dx,$$

and divide by $c_* + 1$. Then we discover that

$$\int_{A(k,\rho')} \Psi(x, Du)\, dx \le \vartheta \int_{A(k,\rho)} \Psi(x, Du)\, dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx, \tag{5.43}$$

where $\vartheta = \frac{c_*}{c_*+1} < 1$, for any $0 < \rho' < \rho < \infty$ with $B_\rho \subset \Omega$.

Now fix $\rho' < \rho$ and consider a sequence

$$\rho_0 := \rho' \quad \text{and} \quad \rho_{i+1} = (1-\lambda)\lambda^i(\rho - \rho') + \rho_i, \quad i = 0, 1, 2, \cdots,$$

where $\lambda \in (0,1)$ is to be chosen later. Applying (5.43) inductively, we obtain from (5.17) that

$$\begin{aligned}
&\int_{A(k,\rho')} \Psi(x, Du)\,dx \\
&\quad \le \vartheta \int_{A(k,\rho_1)} \Psi(x, Du)\,dx + \int_{A(k,\rho_1)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\quad \le \vartheta^2 \int_{A(k,\rho_2)} \Psi(x, Du)\,dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\qquad + \vartheta \int_{A(k,\rho_2)} \Psi\left(x, \frac{u-k}{(1-\lambda)\lambda(\rho-\rho')}\right) dx \\
&\quad \le \vartheta^2 \int_{A(k,\rho_2)} \Psi(x, Du)\,dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\qquad + \Delta_2(\Psi)\vartheta\lambda^{-\log_2 \Delta_2(\Psi)} \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\quad \le \vartheta^i \int_{A(k,\rho_i)} \Psi(x, Du)\,dx \\
&\qquad + \Delta_2(\Psi) \sum_{j=0}^{i-1} \left(\vartheta\lambda^{-\log_2 \Delta_2(\Psi)}\right)^j \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\quad \le \vartheta^i \int_{A(k,\rho_i)} \Psi(x, Du)\,dx \\
&\qquad + \frac{\Delta_2(\Psi)}{(1-\lambda)^{\log_2 \Delta_2(\Psi)}} \sum_{j=0}^{i-1} \left(\vartheta\lambda^{-\log_2 \Delta_2(\Psi)}\right)^j \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx.
\end{aligned}$$

Finally, choosing $\lambda = \lambda(Q, \Delta_2(\Psi)) = \lambda(Q, \Delta_2(G), \Delta_2(H)) \in (0,1)$ in such a way that $\vartheta\lambda^{-\log_2 \Delta_2(\Psi)} < 1$ and passing to the limit for $i \to \infty$, we get

$$\int_{A(k,\rho')} \Psi(x, Du)\,dx$$

$$\leq \frac{\Delta_2(\Psi)}{(1-\lambda)^{\log_2 \Delta_2(\Psi)}(1-\vartheta\lambda^{-\log_2 \Delta_2(\Psi)})} \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx,$$

which proves the lemma. □

For the Hölder continuity of local quasi-minimizers of $\mathcal{F}$, we assume that the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying

$$\limsup_{\rho\to 0+} \omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty, \tag{5.44}$$

or, in other words

$$\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \leq L \quad \text{for every } 0 < \rho \leq 1, \tag{5.45}$$

for some $L > 0$.

We remark that when $(G(t), H(t)) = (t^p, t^q)$ with $1 < p < q$, and $a(\cdot) \in C^{0,\alpha}(\Omega)$ with $\alpha \in (0, 1]$, the condition (5.44) is equivalent to $q \leq p + \alpha$. In addition, when $(G(t), H(t)) = (t^p, t^p \ln(1 + t))$ with $p > 1$, the condition (5.44) is equivalent to

$$\limsup_{\rho\to 0+} \omega(\rho) \ln\left(\frac{1}{\rho}\right) < \infty.$$

Therefore, the condition (5.44) agrees with the classical ones essentially used in [7, 8, 38, 39].

In addition, the condition (5.45) ensures that quasi-minimizers of $\mathcal{F}$ satisfy the following Caccioppoli type inequality provided the modulating coefficient $a(\cdot)$ is suitably small in the right scale.

Lemma 5.3.5 (Almost standard Caccioppoli inequality)**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_r \subset \Omega$ be a ball with $r \leq 1$. Suppose that*

$$\sup_{x\in B_r} a(x) \leq 4\omega(r). \tag{5.46}$$

Then for every $\frac{r}{2} \le r_1 < r_2 \le r$ *and* $k \in \mathbb{R}$ *with* $|k| \le \|u\|_{L^\infty(B_r)}$,

$$\fint_{B_{r_1}} G\left(|D(u-k)_\pm|\right) dx \le c\left(\frac{r}{r_2-r_1}\right)^{c_G+c_H+2} \fint_{B_{r_2}} G\left(\frac{(u-k)_\pm}{r}\right) dx \tag{5.47}$$

holds for some constant $c = c(Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) > 0$.

Proof. If follows from Lemma 5.1.2, Lemma 5.1.7, Lemma 5.3.4, (5.46) and (5.13) that

$$\begin{aligned}
&\int_{B_{r_1}} G\left(|D(u-k)_\pm|\right)\, dx \le \int_{B_{r_1}} \Psi(x, D(u-k)_\pm)\, dx \\
&\le c\int_{B_{r_2}} \Psi\left(x, \frac{(u-k)_\pm}{r_2-r_1}\right) dx \\
&= c\int_{B_{r_2}} \left(1 + a(x)\left(\frac{H}{G}\right)\left(\frac{(u-k)_\pm}{r_2-r_1}\right)\right) G\left(\frac{(u-k)_\pm}{r_2-r_1}\right) dx \\
&\le c\int_{B_{r_2}} \left(1 + \omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right)\right) G\left(\frac{(u-k)_\pm}{r}\,\frac{r}{r_2-r_1}\right) dx \\
&\le c\left(\frac{r}{r_2-r_1}\right)^{c_G+1} \left(1 + \omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right)\right) \int_{B_{r_2}} G\left(\frac{(u-k)_\pm}{r}\right) dx.
\end{aligned}$$

We observe from Lemma 5.1.7, (5.13) and (5.45) that

$$\begin{aligned}
\omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right) &\le \omega(r)\left(\frac{H}{G}\right)\left(\frac{2(\|u\|_{L^\infty(B_r)}+1)r}{r_2-r_1}\cdot\frac{1}{r}\right) \\
&\le \omega(r)\left(\frac{2(\|u\|_{L^\infty(B_r)}+1)r}{r_2-r_1}\right)^{c_H+1}\left(\frac{H}{G}\right)\left(\frac{1}{r}\right) \\
&\le c\left(\frac{r}{r_2-r_1}\right)^{c_H+1} \omega(r)\frac{H(r^{-1})}{G(r^{-1})} \\
&\le c\left(\frac{r}{r_2-r_1}\right)^{c_H+1} L,
\end{aligned}$$

which completes the proof. $\square$

Lemma 5.3.6. *Under the assumptions of Lemma 5.3.5, we further suppose that the density condition*

$$\left|\left\{x \in B_{\frac{r}{2}} : u(x) > \sup_{B_r} u - \frac{1}{2}\operatorname*{osc}_{B_r} u\right\}\right| \le \frac{1}{2}\left|B_{\frac{r}{2}}\right|. \tag{5.48}$$

holds. Then for any $\tau \in (0,1)$, there exists a large natural number $m \ge 3$ depending on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}$ and τ such that

$$\left|\left\{x \in B_{\frac{r}{2}} : u(x) > \sup_{B_r} u - \frac{1}{2^m}\operatorname*{osc}_{B_r} u\right\}\right| \le \tau\left|B_{\frac{r}{2}}\right|.$$

Proof. Let $m \ge 3$ be a large natural number as selected below. Define for $i = 1, 2, \cdots, m$,

$$k_i := \sup_{B_r} u - \frac{1}{2^i}\operatorname*{osc}_{B_r} u, \quad D_i := A\left(k_i, \frac{r}{2}\right) \setminus A\left(k_{i+1}, \frac{r}{2}\right),$$

and

$$w_i(x) := \begin{cases} k_{i+1} - k_i & \text{if } \ u(x) > k_{i+1}, \\ u(x) - k_i & \text{if } \ k_i < u(x) \le k_{i+1}, \\ 0 & \text{if } \ u(x) \le k_i. \end{cases}$$

We note that $G(w_i) \in W^{1,1}(B_{\frac{r}{2}})$ and $G(w_i) = 0$ in $B_{\frac{r}{2}} \setminus A\left(k_1, \frac{r}{2}\right)$ for all $i = 1, 2, \cdots, m$, and that $\left|B_{\frac{r}{2}} \setminus A\left(k_1, \frac{r}{2}\right)\right| \ge \frac{1}{2}\left|B_{\frac{r}{2}}\right|$. Using Hölder's inequality, Sobolev's inequality and a modified form of Young's inequality (5.8) with $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
&\left|A\left(k_{i+1}, \frac{r}{2}\right)\right| G\left(\frac{k_{i+1} - k_i}{r/2}\right) \\
&\quad \le \int_{A\left(k_i, \frac{r}{2}\right)} G\left(\frac{w_i}{r/2}\right) dx \\
&\quad \le \left|A\left(k_i, \frac{r}{2}\right)\right|^{\frac{1}{n}} \left(\int_{A\left(k_i, \frac{r}{2}\right)} \left[G\left(\frac{w_i}{r/2}\right)\right]^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \\
&\quad \le cr \left(\int_{A\left(k_i, \frac{r}{2}\right)} \left[G\left(\frac{w_i}{r/2}\right)\right]^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}}
\end{aligned}$$

$$
\begin{aligned}
&\leq c \int_{D_i} G'\left(\frac{u-k_i}{r/2}\right) |Du| \, dx \\
&\leq \varepsilon \int_{D_i} G(|Du|) \, dx + c(\varepsilon) \int_{D_i} G\left(\frac{u-k_i}{r/2}\right) dx. \qquad (5.49)
\end{aligned}
$$

It follows from Lemma 5.3.5 that

$$
\begin{aligned}
\int_{D_i} G(|Du|) \, dx &\leq c \int_{A(k_i,r)} G\left(\left|\frac{u-k_i}{r}\right|\right) dx \\
&\leq c \int_{A(k_i,r)} G\left(\frac{1}{2^i r} \operatorname*{osc}_{B_r} u\right) dx = c \, G\left(\frac{k_{i+1}-k_i}{r/2}\right) |A(k_i,r)| \\
&\leq c \, G\left(\frac{k_{i+1}-k_i}{r/2}\right) r^n. \qquad (5.50)
\end{aligned}
$$

Also, it is clear that

$$
\int_{D_i} G\left(\frac{u-k_i}{r/2}\right) dx \leq \int_{D_i} G\left(\frac{k_{i+1}-k_i}{r/2}\right) dx = G\left(\frac{k_{i+1}-k_i}{r/2}\right) |D_i|. \quad (5.51)
$$

Combining (5.49) with (5.50) and (5.51), we see that for $i = 1, 2, \cdots, m-1$,

$$
\left| A\left(k_{m-1}, \frac{r}{2}\right) \right| \leq \left| A\left(k_{i+1}, \frac{r}{2}\right) \right| \leq c\varepsilon r^n + c(\varepsilon) |D_i|.
$$

Summing over i from 1 to $m-1$ yields that

$$
\begin{aligned}
(m-1) \left| A\left(k_{m-1}, \frac{r}{2}\right) \right| &\leq c(m-1)\varepsilon r^n + c(\varepsilon) \left| A\left(k_1, \frac{r}{2}\right) \right| \\
&\leq \left(c(m-1)\varepsilon + c(\varepsilon)\right) r^n
\end{aligned}
$$

and hence

$$
\left| A\left(k_{m-1}, \frac{r}{2}\right) \right| \leq \left(c\varepsilon + \frac{c(\varepsilon)}{m-1}\right) r^n \leq \tau \left| B_{\frac{r}{2}} \right|
$$

by taking sufficiently small $\varepsilon = \varepsilon(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}, \tau) \in (0,1)$ and sufficiently large $m = m(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}, \tau) \in \mathbb{N}$. $\square$

Lemma 5.3.7. *Under the assumptions of Lemma 5.3.6, we further find that there exists a small* $\tau_0 = \tau_0(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) \in (0, 2^{-(n+1)})$ *such that* *if*

$$0 < \nu < \frac{1}{2} \operatorname*{osc}_{B_r} u \quad \text{and} \quad \left| A\left(k_0, \frac{r}{2}\right) \right| \le \tau_0 \left| B_{\frac{r}{2}} \right|, \tag{5.52}$$

where $k_0 := \sup_{B_r} u - \nu$, *then*

$$\sup_{B_{\frac{r}{4}}} u \le k_0 + \frac{\nu}{2} = \sup_{B_r} u - \frac{\nu}{2}. \tag{5.53}$$

Proof. We first set the sequences

$$\rho_i := \frac{r}{4}\left(1 + \frac{1}{2^i}\right) \quad \text{and} \quad k_i := k_0 + \frac{\nu}{2}\left(1 - \frac{1}{2^i}\right), \quad i = 0, 1, 2, \cdots,$$

and define

$$D_{i+1} := A(k_i, \rho_{i+1}) \setminus A(k_{i+1}, \rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|A(k_i, \rho_i)|}{|B_{\frac{r}{2}}|}.$$

We note from the definitions of k_i that $(u - k_i)_+ \le \nu \le \|u\|_{L^\infty(B_r)}$. Then we discover from (5.47) and (5.52) that

$$\begin{aligned} \int_{A(k_i, \rho_{i+1})} G(|Du|)\, dx &\le c 2^{(i+3)(c_G + c_H + 2)} \int_{A(k_i, \rho_i)} G\left(\frac{(u - k_i)_+}{r}\right) dx \\ &\le c 2^{i(c_G + c_H + 2)} G\left(\frac{\nu}{r}\right) |A(k_i, \rho_i)|. \end{aligned}$$

It follows from the convexity of G that

$$\begin{aligned} G\left(\fint_{D_{i+1}} |Du|\, dx\right) &\le \fint_{D_{i+1}} G(|Du|)\, dx \\ &\le c 2^{i(c_G + c_H + 2)} \frac{|A(k_i, \rho_i)|}{|D_{i+1}|} G\left(\frac{\nu}{r}\right) \\ &\le G\left(c 2^{i(c_G + c_H + 2)} \frac{|A(k_i, \rho_i)|}{|D_{i+1}|} \frac{\nu}{r}\right). \end{aligned}$$

Therefore, we obtain

$$\fint_{D_{i+1}} |Du|\, dx \le c 2^{i(c_G + c_H + 2)} \frac{|A(k_i, \rho_i)|}{|D_{i+1}|} \frac{\nu}{r}.$$

On the other hand, using Lemma 2.3.3 and the fact that $\tau_0 \in (0, 2^{-(n+1)})$, we have

$$\begin{aligned}\int_{D_{i+1}} |Du|\,dx &\geq c(k_{i+1}-k_i)|A(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}|B_{\rho_{i+1}} \setminus A(k_i,\rho_{i+1})|\rho_{i+1}^{-n}\\ &\geq c2^{-i}\nu|A(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\left(\left|B_{\frac{r}{4}}\right| - \tau_0\left|B_{\frac{r}{2}}\right|\right)r^{-n}\\ &\geq c2^{-i}\nu|A(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\\ &\geq c2^{-i}\nu r^{n-1}Y_{i+1}^{1-\frac{1}{n}}.\end{aligned}$$

Combining these inequalities gives

$$Y_{i+1}^{1-\frac{1}{n}} \leq c2^{i(c_G+c_H+3)}r^{-n}|A(k_i,\rho_i)| \leq c2^{i(c_G+c_H+3)}Y_i,$$

and hence

$$Y_{i+1} \leq c_* 2^{\frac{n(c_G+c_H+3)}{n-1}i}Y_i^{1+\frac{1}{n-1}}$$

for some constant $c_* = c_*(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) > 1$.
Consequently, Lemma 2.3.2 implies that $Y_i \to 0$ as $i \to \infty$, provided

$$Y_0 = \frac{\left|A\left(k_0, \frac{r}{2}\right)\right|}{\left|B_{\frac{r}{2}}\right|} \leq \tau_0 \leq c_*^{-(n-1)}2^{-n(n-1)(c_G+c_H+3)}.$$

Then we obtain

$$\left|A\left(k_0 + \frac{\nu}{2}, \frac{r}{4}\right)\right| = 0,$$

which implies (5.53). $\square$

Proposition 5.3.8. *Under the assumptions of Lemma 5.3.6, let $m \geq 3$ be the natural number determined in Lemma 5.3.6 with $\tau = \tau_0 \in (0, 2^{-(n+1)})$ which is given in Lemma 5.3.7. Then we see that $m \in \mathbb{N}$ depends only on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_r)}$, and we have*

$$\operatorname*{osc}_{B_{\frac{r}{4}}} u \leq \left(1 - \frac{1}{2^{m+1}}\right)\operatorname*{osc}_{B_r} u. \tag{5.54}$$

Proof. We assume without loss of generality that the density condition (5.48) holds; if this is not true, we just consider $-u$ instead of u. Setting

$$\nu := \frac{1}{2^m} \operatorname*{osc}_{B_r} u,$$

we see from Lemma 5.3.6 that (5.52) holds true. Therefore, by Lemma 5.3.7, we have

$$\sup_{B_{\frac{r}{4}}} u \leq \sup_{B_r} u - \frac{\nu}{2} = \sup_{B_r} u - \frac{1}{2^{m+1}} \operatorname*{osc}_{B_r} u,$$

and hence

$$\operatorname*{osc}_{B_r} u \leq 2^{m+1} \left(\sup_{B_r} u - \sup_{B_{\frac{r}{4}}} u \right) \leq 2^{m+1} \left(\operatorname*{osc}_{B_r} u - \operatorname*{osc}_{B_{\frac{r}{4}}} u \right),$$

which implies (5.54). □

The following lemma provides the Hölder continuity of quasi-minimizers of the functional

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{F}_0(v, \Omega) := \int_\Omega [G(|Dv|) + a_0 H(|Dv|)]\, dx, \tag{5.55}$$

where $0 \leq a_0 \leq \|a\|_{L^\infty(\Omega)}$ is a fixed constant. For simplicity, we set

$$\Psi_0(t) := G(t) + a_0 H(t) \tag{5.56}$$

for $t \geq 0$.

Lemma 5.3.9. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}_0$. Then there exist $\beta_0 \in (0,1)$ and $c > 0$, both depending on n, Q, c_G, c_H, but independent of a_0 and u, such that for any fixed ball $B_{r_0} \Subset \Omega$,*

$$\operatorname*{osc}_{B_r} u \leq c \left(\frac{r}{r_0} \right)^{\beta_0} \operatorname*{osc}_{B_{r_0}} u \tag{5.57}$$

holds for every $0 < r \leq r_0$.

Proof. We claim that

$$\frac{1}{c_G + c_H} \leq \frac{t \Psi_0''(t)}{\Psi_0'(t)} \leq c_G + c_H, \quad \forall t > 0. \tag{5.58}$$

Indeed, if $a_0 = 0$, then $\Phi_0(t) \equiv G(t)$ and hence (5.58) holds from (5.30) clearly. If $a_0 > 0$, then we obtain from (5.30) that

$$\begin{aligned}\frac{t\Psi_0''(t)}{\Psi_0'(t)} &= \frac{tG''(t)}{G'(t)+a_0H'(t)} + \frac{ta_0H''(t)}{G'(t)+a_0H'(t)}\\ &\leq \frac{tG''(t)}{G'(t)} + \frac{ta_0H''(t)}{a_0H'(t)}\\ &= \frac{tG''(t)}{G'(t)} + \frac{tH''(t)}{H'(t)} \leq c_G + c_H,\end{aligned}$$

and that

$$\begin{aligned}\frac{\Psi_0'(t)}{t\Psi_0''(t)} &= \frac{G'(t)}{tG''(t)+ta_0H''(t)} + \frac{a_0H'(t)}{tG''(t)+ta_0H''(t)}\\ &\leq \frac{G'(t)}{tG''(t)} + \frac{a_0H'(t)}{ta_0H''(t)}\\ &= \frac{G'(t)}{tG''(t)} + \frac{H'(t)}{tH''(t)} \leq c_G + c_H.\end{aligned}$$

Combining these inequalities yields (5.58).
We now note from Theorem 5.3.3 that u is locally bounded in Ω. Therefore, the result (5.57) follows from [88, Section 6]. □

We are now ready to prove the Hölder continuity of quasi-minimizers of $\mathcal{F}$.

Theorem 5.3.10 (Hölder continuity)**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$ under the assumption (5.45). Then for every open subset $\Omega' \Subset \Omega$ there exists $\beta \in (0,1)$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(\Omega')}$, such that*

$$u \in C^{0,\beta}_{\mathrm{loc}}(\Omega').$$

Proof. We shall show that for a fixed ball $B_{8r_0} \subset \Omega'$ with $8r_0 \leq 1$, there holds

$$\operatorname*{osc}_{B_r} u \leq c\left(\frac{r}{r_0}\right)^{\beta} \operatorname*{osc}_{B_{r_0}} u, \quad \forall r \in (0, r_0], \tag{5.59}$$

for some positive constant c depending only on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(\Omega')}$.

Let us define

$$\mathcal{J} := \left\{ i \in \mathbb{N}_0 : (5.46) \text{ does not hold for } r = \frac{r_0}{4^i} \right\},$$

and

$$j := \begin{cases} \min \mathcal{J} & \text{if} \quad \mathcal{J} \neq \emptyset, \\ \infty & \text{if} \quad \mathcal{J} = \emptyset. \end{cases}$$

If $j \geq 1$, then (5.46) is satisfied in $B_{4^{-i}r_0}$ for $i = 0, \cdots, j-1$. By Proposition 5.3.8, for each $r = 4^{-i}r_0$, $i = 0, \cdots, j-1$, we have

$$\operatorname*{osc}_{B_{\frac{r}{4}}} u \leq \left(1 - \frac{1}{2^{m+1}}\right) \operatorname*{osc}_{B_r} u.$$

This gives

$$\operatorname*{osc}_{B_r} u \leq 4 \left(\frac{r}{r_0}\right)^{\beta_1} \operatorname*{osc}_{B_{r_0}} u \tag{5.60}$$

for every $r \in (4^{-(j+1)}r_0, r_0]$, where

$$\beta_1 = \log_4 \left(\frac{2^{m+1}}{2^{m+1} - 1}\right) \in (0, 1).$$

If $j = \infty$, then (5.60) holds for every $r \in (0, r_0]$, which is the desired conclusion (5.59) with $\beta = \beta_1$. In the case $1 \leq j < \infty$, we claim that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{4^{-j}r_0}} [G(|Dv|) + a_0 H(|Dv|)]\, dx, \quad a_0 = \sup_{B_{4^{-j}r_0}} a(\cdot). \tag{5.61}$$

Indeed, since (5.46) does not hold for $r = 4^{-j}r_0$, there exists $x_M \in \overline{B}_{4^{-j}r_0}$ such that $a(x_M) = a_0 > 4\omega(4^{-j}r_0)$. Then for every $x \in B_{4^{-j}r_0}$,

$$a(x_M) - a(x) \leq \omega(2 \cdot 4^{-j}r_0) \leq 2\omega(4^{-j}r_0),$$

and hence

$$a_0 \leq 2a_0 - 4\omega(4^{-j}r_0) \leq 2a(x) \leq 2a_0.$$

Since $\Psi(x, Du) \in L^1(B_{4^{-j}r_0})$, it follows that

$$G(|Dv|) + a_0 H(|Dv|) \in L^1(B_{4^{-j}r_0}).$$

Furthermore, one can check that u is a $(2Q)$-minimizer of the functional (5.61). Now, Lemma 5.3.9 gives

$$\operatorname*{osc}_{B_r} u \leq c\left(\frac{r}{4^{-j}r_0}\right)^{\beta_0} \operatorname*{osc}_{B_{4^{-j}r_0}} u \tag{5.62}$$

for every $r \in (0, 4^{-j}r_0]$. Here, $\beta_0 \in (0,1)$ and $c > 0$ both depends only on n, Q, c_G, c_H. Combining (5.60) and (5.62), we conclude that (5.59) holds for $\beta = \min\{\beta_0, \beta_1\}$. Finally, if $j = 0$, then we see at once that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{r_0}} [G(|Dv|) + a_0 H(|Dv|)]\, dx, \quad a_0 = \sup_{B_{r_0}} a(\cdot),$$

and hence we have the desired conclusion (5.59) with $\beta = \beta_0$. □

Remark 5.3.11. *As already mentioned at the end of Section 5.1, we assume (5.29) and (5.30) throughout the chapter. In particular, the boundedness of quasi-minimizers of $\mathcal{F}$ is proved in Theorem 5.3.3, assuming that (5.29) holds. Once the local boundedness of quasi-minimizers has been obtained, Theorem 5.3.10 holds without the assumption (5.29). Therefore, for a locally bounded quasi-minimizer $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ of $\mathcal{F}$, we can prove the Hölder continuity of u under the only assumptions (5.30) and (5.45).*

Remark 5.3.12. *Our condition (5.44) provides a characterization of the modulating coefficient $a(\cdot)$. More precisely, a modulus of continuity of $a(\cdot)$ is exactly calibrated to the size of the phase transition. For example, it is evident that the natural assumption for the modulating coefficient in the functional*

$$v \in W^{1,1}(\Omega) \mapsto \int_{\Omega} [|Dv|^p + a(x)|Dv|^p [\ln(1+|Dv|)]^{\gamma}]\, dx$$

with $p > 1$ and $\gamma > 0$, is

$$\limsup_{\rho \to 0+} \omega(\rho) \left[\ln\left(\frac{1}{\rho}\right)\right]^{\gamma} < \infty.$$

Similarly, for the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_{\Omega} [|Dv|^p + a(x)|Dv|^p \ln\ln(e+|Dv|)]\, dx$$

with $p > 1$, the natural assumption for the modulating coefficient is

$$\limsup_{\rho \to 0+} \omega(\rho) \ln \ln \left(\frac{1}{\rho} \right) < \infty.$$

5.4 The Harnack inequality

In this section, we prove the Harnack inequality for quasi-minimizers of $\mathcal{F}$. The following lemma provides the weak Harnack inequality of quasi-minimizers of the functional $\mathcal{F}_0$ in (5.55), see [88].

Lemma 5.4.1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}_0$ and let $B \Subset \Omega$ be a ball. Then for any exponent $q_+ > 0$ and every $0 < t < s < 1$, we have*

$$\sup_{tB} |u| \leq c^* \left(\fint_{sB} |u|^{q_+} \, dx \right)^{\frac{1}{q_+}} \tag{5.63}$$

for some constant $c^* = c^*(n, Q, c_G, c_H, s - t, q_+) > 1$. Moreover, if u is non-negative, then there exists an exponent $q_- = q_-(n, Q, c_G, c_H) \in (0, 1)$ such that for every $t, s \in (0, 1)$,

$$\inf_{tB} u \geq \frac{1}{c_*} \left(\fint_{sB} u^{q_-} \, dx \right)^{\frac{1}{q_-}} \tag{5.64}$$

holds for some constant $c_* = c_*(n, Q, c_G, c_H, t, s) > 1$.

Lemma 5.4.2. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a non-negative Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{3r} \subset \Omega$ be a ball with $3r \leq 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) \leq 12\omega(r). \tag{5.65}$$

For any $\tau_1, \tau_2 \in (0, 1)$, there exists a large natural number m depending on $n, Q, c_G,$ $c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau_1$ and τ_2 such that for any $0 < \lambda \leq \|u\|_{L^\infty(B_{3r})}$ if

$$|\{x \in B_r : u(x) \geq \lambda\}| \geq \tau_1 |B_r| \tag{5.66}$$

holds, then

$$\left| \left\{ x \in B_{2r} : u(x) \leq 2^{-m} \lambda \right\} \right| \leq \tau_2 |B_{2r}|. \tag{5.67}$$

Proof. Let $m \geq 3$ be a large natural number to be determined later. Define for $i = 1, 2, \cdots, m$,

$$k_i := 2^{-i}\lambda, \quad D_i^- := A^-(k_i, 2r) \setminus A^-(k_{i+1}, 2r)$$

and

$$w_i(x) := \begin{cases} 0 & \text{if } u(x) > k_i, \\ k_i - u(x) & \text{if } k_{i+1} < u(x) \leq k_i, \\ k_i - k_{i+1} & \text{if } u(x) \leq k_{i+1}. \end{cases}$$

Then $G(w_i) \in W^{1,1}(B_{2r})$ and $G(w_i) = 0$ in $B_{2r} \setminus A^-(k_1, 2r)$ for all $i = 1, 2, \cdots, m$. We note from (5.66) that

$$\begin{aligned} \left|B_{2r} \setminus A^-(k_1, 2r)\right| &= |\{x \in B_{2r} : u(x) > k_1\}| \\ &\geq |\{x \in B_r : u(x) \geq \lambda\}| \geq \tau_1 |B_r| = 2^{-n}\tau_1 |B_{2r}|. \end{aligned}$$

Using Hölder's inequality, Sobolev's inequality and a modified form of Young's inequality (5.8) with $\varepsilon \in (0, 1)$, we have

$$\begin{aligned} &|A^-(k_{i+1}, 2r)| G\left(\frac{k_i - k_{i+1}}{2r}\right) \\ &\quad \leq \int_{A^-(k_i,2r)} G\left(\frac{w_i}{2r}\right) dx \\ &\quad \leq |A^-(k_i, 2r)|^{\frac{1}{n}} \left(\int_{A^-(k_i,2r)} \left[G\left(\frac{w_i}{2r}\right)\right]^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \\ &\quad \leq cr \left(\int_{A^-(k_i,2r)} \left[G\left(\frac{w_i}{2r}\right)\right]^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \\ &\quad \leq c(n, \tau_1) \int_{D_i^-} G'\left(\frac{k_i - u}{2r}\right) |Du|\, dx \\ &\quad \leq \varepsilon \int_{D_i^-} G(|Du|)\, dx + c(\varepsilon) \int_{D_i^-} G\left(\frac{k_i - u}{2r}\right) dx, \end{aligned} \tag{5.68}$$

where $c(\varepsilon) \equiv c(n, \tau_1, \varepsilon)$ is a positive constant. It follows from Caccioppoli inequality (5.47) that

$$\int_{D_i^-} G(|Du|)\, dx \leq \int_{B_{2r}} G(|D(u - k_i)_-|)\, dx$$

$$
\begin{aligned}
&\le c \int_{B_{3r}} G\left(\frac{(u-k_i)_-}{3r}\right) dx \\
&\le c \int_{B_{3r}} G\left(\frac{2^{-i}\lambda}{3r}\right) dx = c\, G\left(\frac{k_i - k_{i+1}}{3r/2}\right) |B_{3r}| \\
&\le c\, G\left(\frac{2(k_i - k_{i+1})}{2r}\right) r^n \le c\, G\left(\frac{k_i - k_{i+1}}{2r}\right) r^n. \qquad (5.69)
\end{aligned}
$$

Also, it is clear that

$$
\int_{D_i^-} G\left(\frac{k_i - u}{2r}\right) dx \le \int_{D_i^-} G\left(\frac{k_i - k_{i+1}}{2r}\right) dx = G\left(\frac{k_i - k_{i+1}}{2r}\right) |D_i^-|. \tag{5.70}
$$

Combining (5.68) with (5.69) and (5.70), we see that for $i = 1, 2, \cdots, m-1$,

$$
|A^-(k_{m-1}, 2r)| \le |A^-(k_{i+1}, 2r)| \le c\varepsilon r^n + c(\varepsilon)|D_i^-|.
$$

Summing over i from 1 to $m-1$ yields that

$$
\begin{aligned}
(m-1)|A^-(k_{m-1}, 2r)| &\le c(m-1)\varepsilon r^n + c(\varepsilon)|A^-(k_1, 2r)| \\
&\le \left(c(m-1)\varepsilon + c(\varepsilon)\right) r^n
\end{aligned}
$$

and hence

$$
|A^-(k_{m-1}, 2r)| \le \left(c\varepsilon + \frac{c(\varepsilon)}{m-1}\right) r^n \le \tau_2 |B_{2r}|,
$$

if we take sufficiently small $\varepsilon = \varepsilon(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau_2) \in (0,1)$ and sufficiently large $m = m(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau_1, \tau_2) \in \mathbb{N}$. □

Proposition 5.4.3. *Let the assumptions in Lemma 5.4.2 hold. For any $\tau \in (0,1)$, there exists a small $\delta_1 = \delta_1(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that for any $0 < \lambda \le \|u\|_{L^\infty(B_{3r})}$, if*

$$
|\{x \in B_r : u(x) \ge \lambda\}| \ge \tau |B_r| \tag{5.71}
$$

holds, then

$$
\inf_{B_r} u \ge \delta_1 \lambda. \tag{5.72}
$$

Proof. It suffices to prove the proposition for $\tau \in (0, 2^{-(n+1)})$. We fix $m_0 \in \mathbb{N}$, and set the sequences

$$\rho_i := r\left(1+\frac{1}{2^i}\right) \quad \text{and} \quad k_i := \left(\frac{1}{2}+\frac{1}{2^i}\right)2^{-m_0}\lambda, \quad i=0,1,2,\cdots.$$

We also define

$$D_{i+1}^- := A^-(k_i,\rho_{i+1}) \setminus A^-(k_{i+1},\rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|A^-(k_i,\rho_i)|}{|B_{\rho_i}|}.$$

Since u is non-negative, we have $(u-k_i)_- \leq 2^{-m_0}\lambda$. By (5.47), we get

$$\begin{aligned}\int_{A^-(k_i,\rho_{i+1})} G(|Du|)\,dx &\leq c2^{(i+3)(c_G+c_H+2)}\int_{A^-(k_i,\rho_i)} G\left(\frac{(u-k_i)_-}{2r}\right)dx\\ &\leq c2^{i(c_G+c_H+2)}G\left(\frac{2^{-m_0}\lambda}{r}\right)|A^-(k_i,\rho_i)|.\end{aligned}$$

We deduce from the convexity of G that

$$\begin{aligned}G\left(\fint_{D_{i+1}^-}|Du|\,dx\right) &\leq \fint_{D_{i+1}^-} G(|Du|)\,dx\\ &\leq c2^{i(c_G+c_H+2)}\frac{|A^-(k_i,\rho_i)|}{|D_{i+1}^-|}G\left(\frac{2^{-m_0}\lambda}{r}\right)\\ &\leq G\left(c2^{i(c_G+c_H+2)}\frac{|A^-(k_i,\rho_i)|}{|D_{i+1}^-|}\frac{2^{-m_0}\lambda}{r}\right).\end{aligned}$$

Therefore, we obtain

$$\fint_{D_{i+1}^-}|Du|\,dx \leq c2^{i(c_G+c_H+2)}\frac{|A^-(k_i,\rho_i)|}{|D_{i+1}^-|}\frac{2^{-m_0}\lambda}{r}.$$

On the other hand, using Lemma 2.3.4 and the fact that $\tau \in (0,2^{-(n+1)})$, we have

$$\begin{aligned}\int_{D_{i+1}^-}|Du|\,dx &\geq c(k_i-k_{i+1})|A^-(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}|B_{\rho_{i+1}} \setminus A^-(k_i,\rho_{i+1})|\rho_{i+1}^{-n}\\ &\geq c2^{-i}\cdot 2^{-m_0}\lambda|A^-(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\left(|B_{2r}|-\tau|B_r|\right)r^{-n}\\ &\geq c2^{-i}\cdot 2^{-m_0}\lambda|A^-(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\end{aligned}$$

$$\geq c2^{-i} \cdot 2^{-m_0} \lambda r^{n-1} Y_{i+1}^{1-\frac{1}{n}}.$$

Combining these inequalities gives

$$Y_{i+1}^{1-\frac{1}{n}} \leq c2^{i(c_G+c_H+3)} r^{-n} |A^-(k_i, \rho_i)| \leq c2^{i(c_G+c_H+3)} Y_i,$$

and hence

$$Y_{i+1} \leq c_0 2^{\frac{n(c_G+c_H+3)}{n-1} i} Y_i^{1+\frac{1}{n-1}}$$

for some constant $c_0 = c_0(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}) > 1$. Here we note from Lemma 5.4.2 that there exists a large natural number m_0 depending only on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}$ such that

$$\left|\left\{x \in B_{2r} : u(x) \leq 2^{-m_0}\lambda\right\}\right| \leq c_0^{-(n-1)} 2^{-n(n-1)(c_G+c_H+3)} |B_{2r}|.$$

Then it is clear that

$$Y_0 = \frac{|A^-(k_0, 2r)|}{|B_{2r}|} = \frac{|\{x \in B_{2r} : u(x) \leq 2^{-m_0}\lambda\}|}{|B_{2r}|} \leq c_0^{-(n-1)} 2^{-n(n-1)(c_G+c_H+3)},$$

and hence $Y_i \to 0$ as $i \to \infty$ by Lemma 2.3.2. Consequently, we obtain

$$\left|A^-\left(2^{-(m_0+1)}\lambda, r\right)\right| = 0,$$

which implies (5.72) with $\delta_1 = 2^{-(m_0+1)}$. □

Proposition 5.4.4. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a non-negative Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{3r} \subset \Omega$ be a ball with $3r \leq 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) > 12\omega(r). \tag{5.73}$$

For any $\tau \in (0,1)$, there exists $\delta_2 = \delta_2(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that if

$$|\{x \in B_r : u(x) \geq \lambda\}| \geq \tau |B_r| \tag{5.74}$$

for $0 < \lambda \leq \|u\|_{L^\infty(B_{3r})}$, then

$$\inf_{B_r} u \geq \delta_2 \lambda. \tag{5.75}$$

Proof. By (5.73), there exists $x_M \in \overline{B}_{3r}$ such that $a(x_M) = a_0 > 12\omega(r)$. Then for every $x \in B_{3r}$,

$$a(x_M) - a(x) \leq \omega(6r) \leq 6\omega(r),$$

and hence

$$a_0 \leq 2a_0 - 12\omega(r) \leq 2a(x) \leq 2a_0.$$

Since $\Psi(x, Du) \in L^1(B_{3r})$, it follows that

$$G(|Dv|) + a_0 H(|Dv|) \in L^1(B_{3r}).$$

Furthermore, one can see that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{3r}} [G(|Dv|) + a_0 H(|Dv|)]\, dx, \quad a_0 = \sup_{B_{3r}} a(\cdot).$$

Now, using (5.64) in Lemma 5.4.1 with $B \equiv B_{3r}$ and $t = s = \frac{1}{3}$, we see from (5.74) that

$$\inf_{B_r} u \geq \frac{\tau^{\frac{1}{q_-}} \lambda}{c_*},$$

which implies (5.75) with $\delta_2 := \tau^{\frac{1}{q_-}} c_*^{-1}$. □

An immediate consequence of Proposition 5.4.3 and 5.4.4 is the following.

Corollary 5.4.5. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a non-negative Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{3r} \subset \Omega$ be a ball with $3r \leq 1$. For any $\tau \in (0, 1)$, there exists a small $\delta = \delta(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that if*

$$|\{x \in B_r : u(x) \geq \lambda\}| \geq \tau |B_r|$$

for $0 < \lambda \leq \|u\|_{L^\infty(B_{3r})}$, then

$$\inf_{B_r} u \geq \delta\lambda.$$

From Corollary 5.4.5 and the covering arguments in [75, Section 7], we obtain the following weak Harnack inequality for quasi-minimizers of $\mathcal{F}$. For the proof we refer the reader to [7, Theorem 3.5] and [69, Theorem 5.7].

Theorem 5.4.6 (The weak Harnack inequality)**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a non-negative Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{9r} \equiv B_{9r}(x_0) \subset \Omega$ with $9r \leq 1$. Then there exists an exponent $q_- > 0$ and a constant $c > 1$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{9r})}$, such that*

$$\inf_{B_r} u \geq \frac{1}{c} \left(\fint_{B_{2r}} u^{q_-} \, dx \right)^{\frac{1}{q_-}} . \tag{5.76}$$

To prove the sup-estimate for quasi-minimizers of $\mathcal{F}$, we now introduce the scaled functions and the corresponding functional. Let us define, for $R \in (0,1]$ and $r > 0$ with $B_r \Subset \Omega$,

$$u_R(x) := \frac{u(Rx)}{R}, \quad a_R(x) := a(Rx) \quad (x \in B_r)$$

and

$$\mathcal{F}_R(v, K) := \int_K [G(|Dv|) + a_R(x) H(|Dv|)] \, dx \quad (K \Subset B_r).$$

Lemma 5.4.7. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$. Let $R \in (0,1]$ and suppose that $B_r \Subset \Omega$. Then u_R is a Q-minimizer of $\mathcal{F}_R$ in B_r.*

Proof. We first observe that $Du_R(x) = Du(Rx)$. Since $B_r \Subset \Omega$, we see that $\mathcal{F}(u, B_r) < +\infty$, and hence

$$\begin{aligned}
\mathcal{F}_R(u_R, B_r) &= \int_{B_r} [G(|Du(Rx)|) + a(Rx) H(|Du(Rx)|)] \, dx \\
&= \frac{1}{R^n} \int_{B_{Rr}} [G(|Du(y)|) + a(y) H(|Du(y)|)] \, dy \\
&\leq \frac{1}{R^n} \int_{B_r} [G(|Du(y)|) + a(y) H(|Du(y)|)] \, dy \\
&= \frac{1}{R^n} \mathcal{F}(u, B_r) < +\infty.
\end{aligned}$$

Furthermore, for any $v_R \in W^{1,1}_{\mathrm{loc}}(B_r)$ with $K := \mathrm{supp}(u_R - v_R) \Subset B_r$, we have

$$\mathrm{supp}(u - v) = \{Rx : x \in K\} =: RK,$$

and

$$\begin{aligned}\mathcal{F}_R(u_R, K) &= \int_K [G(|Du(Rx)|) + a(Rx)H(|Du(Rx)|)]\,dx \\ &= \frac{1}{R^n}\int_{RK} [G(|Du(y)|) + a(y)H(|Du(y)|)]\,dy \\ &\le \frac{Q}{R^n}\int_{RK} [G(|Dv(y)|) + a(y)H(|Dv(y)|)]\,dy \\ &= Q\int_K [G(|Dv(Rx)|) + a(Rx)H(|Dv(Rx)|)]\,dx = Q\mathcal{F}_R(v_R, K).\end{aligned}$$

Therefore, u_R is a Q-minimizer of $\mathcal{F}_R$ in B_r. □

From the definition of the scaled function $a_R(\cdot)$, one can directly obtain the following lemma.

Lemma 5.4.8. *Let $R \in (0,1]$ and suppose that $B_{4r} \subset B_1 \subset \Omega$. Then the function $a_R : B_{\frac{1}{R}} \to [0,\infty)$ has a modulus of continuity ω_R satisfying*

$$\omega_R(\rho) = \omega(R\rho) \quad \textit{for all} \;\; 0 < \rho \le \frac{1}{R}.$$

Moreover, we have

$$\sup_{x\in B_{3r}} a(x) \le 12\omega(r) \quad \Longleftrightarrow \quad \sup_{x\in B_{\frac{3r}{R}}} a_R(x) \le 12\omega_R\left(\frac{r}{R}\right).$$

We now prove the sup-estimate for quasi-minimizers of $\mathcal{F}$. For this, we consider two cases separately, as in the proof of the weak Harnack inequality.

Proposition 5.4.9. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{4r} \subset \Omega$ be a ball with $4r \le 1$. Suppose that*

$$\sup_{x\in B_{3r}} a(x) \le 12\omega(r).$$

Then for any exponent $q_+ > 0$, we have the estimate

$$\sup_{B_r} |u| \le c\left(\fint_{B_{2r}} |u|^{q_+}\,dx\right)^{\frac{1}{q_+}} \tag{5.77}$$

for some constant $c > 1$ depending on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{4r})}$ and q_+.

Proof. Let us consider the scaled functions

$$u_r(x) = \frac{u(rx)}{r}, \quad a_r(x) = a(rx), \quad x \in B_4.$$

Then by Lemma 5.4.7 and Lemma 5.4.8, we see that the Caccioppoli inequality (5.47) holds for u_r. For $1 \le t < s \le 2$, we now set the sequences

$$\rho_i := t + \frac{s-t}{2^i} \quad \text{and} \quad k_i := 2d\left(1 - \frac{1}{2^{i+1}}\right), \quad i = 0, 1, 2, \cdots,$$

where $d > 0$ is to be chosen later. We further define

$$\widetilde{\rho}_i := \frac{\rho_i + \rho_{i+1}}{2} \quad \text{and} \quad Y_i := \frac{1}{G(d)} \int_{A_r(k_i,\rho_i)} G(u_r - k_i)\, dx,$$

where $A_r(k,\rho) := \{x \in B_\rho : u_r > k\}$. Let $\eta_i \in C_0^\infty(B_{\widetilde{\rho}_i})$ be a cut-off function with $0 \le \eta_i \le 1$, $\eta_i \equiv 1$ on $B_{\rho_{i+1}}$, and $|D\eta_i| \le \frac{4}{\rho_i - \rho_{i+1}}$. Using Hölder's inequality, Sobolev's inequality and a modified form of Young's inequality (5.8) with $\varepsilon = 1$, we have

$$\begin{aligned}
G(d)Y_{i+1} &\le \int_{B_{\widetilde{\rho}_i}} G((u_r - k_{i+1})_+ \eta_i)\, dx \\
&\le |A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \left(\int_{B_{\widetilde{\rho}_i}} [G((u_r - k_{i+1})_+ \eta_i)]^{\frac{n}{n-1}}\, dx \right)^{\frac{n-1}{n}} \\
&\le c|A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \\
&\quad \times \int_{B_{\widetilde{\rho}_i}} G'((u_r - k_{i+1})_+ \eta_i)\, [|D(u_r - k_{i+1})_+|\eta_i + (u_r - k_{i+1})_+ |D\eta_i|]\, dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \int_{B_{\widetilde{\rho}_i}} G'((u_r - k_{i+1})_+)|D(u_r - k_{i+1})_+|\, dx \\
&\qquad + c|A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \frac{2^{i+3}}{s-t} \int_{B_{\widetilde{\rho}_i}} G'((u_r - k_{i+1})_+)(u_r - k_{i+1})_+\, dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \left[\int_{B_{\widetilde{\rho}_i}} G(|D(u_r - k_{i+1})_+|)\, dx + \int_{B_{\widetilde{\rho}_i}} G((u_r - k_{i+1})_+)\, dx \right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + c|A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \frac{2^{i+3}}{s-t} \int_{B_{\tilde{\rho}_i}} G((u_r-k_{i+1})_+)\,dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \left[\int_{B_{\tilde{\rho}_i}} G(|D(u_r-k_{i+1})_+|)\,dx \right. \\
&\qquad\qquad \left. + \frac{2^{i+3}}{s-t} \int_{B_{\tilde{\rho}_i}} G((u_r-k_{i+1})_+)\,dx \right] \\
&\le c|A_r(k_{i+1},\rho_i)|^{\frac{1}{n}} \left(\frac{2^{i+3}}{s-t} \right)^{c_G+c_H+2} \int_{B_{\rho_i}} G((u_r-k_{i+1})_+)\,dx.
\end{aligned}
$$

Here we observe from (5.13) that

$$
\begin{aligned}
|A_r(k_{i+1},\rho_i)| &\le \frac{1}{G(k_{i+1}-k_i)} \int_{A_r(k_{i+1},\rho_i)} G(u_r-k_i)\,dx \\
&= \frac{1}{G(d/2^{i+1})} \int_{A_r(k_{i+1},\rho_i)} G(u_r-k_i)\,dx \\
&\le \frac{G(d)}{G(d/2^{i+1})} Y_i \le 2^{(i+1)(c_G+1)} Y_i \le c \left(\frac{2^{i+3}}{s-t} \right)^{c_G+c_H+2} Y_i
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{B_{\rho_i}} G((u_r-k_{i+1})_+)\,dx &= \int_{A_r(k_{i+1},\rho_i)} G(u_r-k_{i+1})\,dx \\
&\le \int_{A_r(k_i,\rho_i)} G(u_r-k_i)\,dx = G(d) Y_i.
\end{aligned}
$$

Combining these inequalities yields

$$
Y_{i+1} \le \frac{c_0}{(s-t)^\kappa} 2^{i\kappa} Y_i^{1+\frac{1}{n}}
$$

for some constant $c_0 > 1$ depending only on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{4r})}$, where $\kappa = \left(1+\frac{1}{n}\right)(c_G+c_H+2) > 1$. Applying Lemma 2.3.2, we have $Y_i \to 0$ as $i \to \infty$, provided

$$
Y_0 = \frac{1}{G(d)} \int_{A_r(d,s)} G(u_r-d)\,dx \le \left[\frac{c_0}{(s-t)^\kappa} \right]^{-n} 2^{-n^2\kappa}. \tag{5.78}
$$

It is clear that (5.78) is satisfied if we choose $d > 0$ such that

$$G(d) = \frac{2^{n^2\kappa} c_0^n}{(s-t)^{n\kappa}} \int_{B_s} G((u_r)_+)\, dx. \tag{5.79}$$

Then we obtain $u_r \le 2d$ in B_t, which together with (5.79) implies

$$G\left(\sup_{B_t}(u_r)_+\right) \le \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G((u_r)_+)\, dx. \tag{5.80}$$

We note from Lemma 5.1.9 that there exists $\gamma = \gamma(c_G) > 1$ such that $t \mapsto G\left(t^{\frac{1}{\gamma}}\right)$ is a concave function. Then it follows from (5.80) and Jensen's inequality that

$$\begin{aligned} G\left(\sup_{B_t}(u_r)_+\right) &\le \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G((u_r)_+)\, dx \\ &= \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G\left(((u_r)_+^\gamma)^{\frac{1}{\gamma}}\right) dx \\ &\le \frac{c}{(s-t)^{n\kappa}} G\left(\left(\fint_{B_s} (u_r)_+^\gamma\, dx\right)^{\frac{1}{\gamma}}\right) \\ &\le G\left(\frac{c}{(s-t)^{n\kappa}} \left(\fint_{B_s} (u_r)_+^\gamma\, dx\right)^{\frac{1}{\gamma}}\right), \end{aligned}$$

and hence

$$\sup_{B_t}(u_r)_+ \le \frac{c}{(s-t)^{n\kappa}} \left(\fint_{B_s} (u_r)_+^\gamma\, dx\right)^{\frac{1}{\gamma}}.$$

Since $-u$ is also a Q-minimizer of $\mathcal{F}$, we get

$$\sup_{B_t}|u_r| \le \frac{c}{(s-t)^{n\kappa}} \left(\fint_{B_s} |u_r|^\gamma\, dx\right)^{\frac{1}{\gamma}}.$$

Moreover, for $0 < q_+ < \gamma$, we obtain from Young's inequality that

$$\sup_{B_t}|u_r| \le \frac{c}{(s-t)^{n\kappa}} \left[\sup_{B_s}|u_r|\right]^{1-\frac{q_+}{\gamma}} \left(\fint_{B_s} |u_r|^{q_+}\, dx\right)^{\frac{1}{\gamma}}$$

$$\leq \frac{1}{2} \sup_{B_s} |u_r| + \frac{c}{(s-t)^{\frac{n\kappa\gamma}{q_+}}} \left(\fint_{B_2} |u_r|^{q_+} \, dx \right)^{\frac{1}{q_+}},$$

as $1 \leq t < s \leq 2$. Then Lemma 2.3.1 with $\phi(t) := \sup_{B_t} |u_r|$ yields

$$\sup_{B_1} |u_r| \leq c \left(\fint_{B_2} |u_r|^{q_+} \, dx \right)^{\frac{1}{q_+}}, \tag{5.81}$$

where c is a positive constant depending on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{4r})}$ and q_+.
On the other hand, the inequality (5.81) also holds for $q_+ \geq \gamma$ by Hölder's inequality. Finally, from the definition of u_r, we obtain the desired conclusion (5.77). □

Proposition 5.4.10. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{4r} \subset \Omega$ be a ball with $4r \leq 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) > 12\omega(r).$$

Then for any exponent $q_+ > 0$, we have the estimate

$$\sup_{B_r} |u| \leq c \left(\fint_{B_{2r}} |u|^{q_+} \, dx \right)^{\frac{1}{q_+}} \tag{5.82}$$

for some constant $c > 1$ depending on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{4r})}$ and q_+.

Proof. As in the proof of Proposition 5.4.4, we see that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{3r}} [G(|Dv|) + a_0 H(|Dv|)] \, dx, \quad a_0 = \sup_{B_{3r}} a(\cdot) > 0.$$

Therefore, (5.63) in Lemma 5.4.1 with $B \equiv B_{3r}$, $t = \frac{1}{3}$ and $s = \frac{2}{3}$ directly gives (5.82). □

Combining Proposition 5.4.9 and 5.4.10 yields the following sup-estimate.

Corollary 5.4.11. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{4r} \subset \Omega$ be a ball with $4r \leq 1$ Then for any exponent*

$q_+ > 0$, we have the estimate

$$\sup_{B_r} |u| \le c \left(\fint_{B_{2r}} |u|^{q_+} \, dx \right)^{\frac{1}{q_+}} \tag{5.83}$$

for some constant $c > 1$ depending on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{4r})}$ and q_+.

Finally, from Theorem 5.4.6 and Corollary 5.4.11 with $q_+ = q_-$, we obtain the Harnack inequality of quasi-minimizers of $\mathcal{F}$.

Theorem 5.4.12 (The Harnack inequality)**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a non-negative Q-minimizer of $\mathcal{F}$ under the assumption (5.45), and let $B_{9r} \subset \Omega$ be a ball with $9r \le 1$. Then there exists a constant $c > 1$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{9r})}$, such that*

$$\sup_{B_r} u \le c \inf_{B_r} u.$$

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국문초록

이 학위논문에서는 비균일 타원형 및 포물형 문제에 대하여 연구한다. 먼저, 우리는 점근적으로 정규적인 문제에 대한 대역적 칼데론-지그문트 이론에 대하여 연구한다. 구체적으로는 점근적으로 정규적인 타원형 방정식, 점근적으로 정규적인 포물형 방정식, 비정칙 장애물을 가진 점근적으로 정규적인 타원형 방정식, 변동 성장조건을 가지는 점근적으로 정규적인 방정식, 점근적으로 선형인 비발산 타원형 방정식, 점근적으로 완전비선형인 타원형 방정식을 다룬다.

또한, 매끄럽지 않은 경계를 가진 유계영역에서 발산형 이중위상문제에 대하여 연구한다. 이중위상문제는 조절 계수에 따라 타원성과 성장 조건이 변한다는 특징이 있으며, 이는 강한 비등방성 물질의 특징을 설명하는 모델이 된다. 본 연구에서는 다항 성장조건과 로그 성장조건을 가지는 이중위상문제의 초함수해에 대하여 대역적 그래디언트 가늠을 얻는다.

마지막으로, 일반화된 이중위상 범함수의 준최소자에 대한 정칙성 결과들을 확립하기 위해 조절 계수에 대한 최적의 조건을 연구한다.

주요어휘: 점근적으로 정규적인 문제, 칼데론-지그문트 이론, 이중위상문제, 비표준 성장, 비균일 타원성, 정칙성
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